THEORY

OF

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THEORY

OF

DIFFERENTIAL EQUATIONS.

PART I.

EXACT EQUATIONS AND PFAFF'S PROBLEM.

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PREFACE.

THE present volume is the first contribution towards the fulfilment of a promise made at the time of publication of my *Treatise on Differential Equations.* My desire has been to include every substantial contribution to the development of the particular subject herein dealt with; and the historical form, into which the treatment has been cast, has facilitated the indication of the continuous course of the development.

All sources of information, which have been drawn upon, are quoted in their proper connection; a few investigations have been added, which I believe to be new; and some examples have been made, in order to provide illustrations of various methods.

In the revision of the proof-sheets I have had, and wish to acknowledge most gratefully, the valuable assistance of my friend Mr. H. M. Taylor, Fellow of Trinity College, Cambridge. The volume owes much to the care and trouble he has ungrudgingly bestowed upon it. My thanks are also due to Mr. H. F. Baker, Fellow of St. John's College, Cambridge, for his kindness in reading the proof-sheets.

A. R. FORSYTH.

TRINITY COLLEGE, CAMBRIDGE,
28 *July*, 1890.

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CHAPTER I.

SINGLE EXACT EQUATION*.

1. WHEN a number of variables $x, y, z, u, \ldots$ are connected by a permanent relation of the form

$$\phi(x, y, z, u, \ldots) = a \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where a is a constant, any simultaneous small variations $dx, dy, dz, du, \ldots$ to which the variables are subjected are so related that the equation

$$\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz + \frac{\partial\phi}{\partial u}du + \ldots = 0$$

is satisfied; and, if a relation between the small variations be given in this form, the equivalent integral relation is at once obtained in the form of the first equation.

The equation which connects these small variations is exact, for its left-hand member is a complete differential. But if the first differential coefficients of ϕ with regard to the variables have a common factor μ so that we may take

$$\frac{\partial\phi}{\partial x} = \mu P, \quad \frac{\partial\phi}{\partial y} = \mu Q, \quad \frac{\partial\phi}{\partial z} = \mu R, \quad \frac{\partial\phi}{\partial u} = \mu S, \ldots\ldots$$

* It is to be understood that the investigations in the first chapter relative to exact linear equations are additional to the very slight sketch of such equations given in §§ 150—164 of my *Treatise on Differential Equations*, hereafter referred to as *Treatise*; and that the investigations in the second chapter relative to the integration of systems of partial differential equations are intended specially to indicate Mayer's theory of a system of equations of a particular form and to be supplementary to the investigations of Bour and Jacobi.

then the relation connecting the variations becomes

$$Pdx + Qdy + Rdz + Sdu + \ldots = 0 \quad \ldots\ldots\ldots\ldots\ldots(2),$$

on the removal of the factor μ which is not dependent on these variations. This new equation (2) is essentially the same as the earlier equation; but it is not necessarily, and in general it is not, an exact equation. In order to be made an exact equation so that the integral relation (1) may be deduced, the factor μ, which may be called the integrating factor, must be restored; and, as no indication of the form of μ survives in the reduced equation, the determination of the factor must be made by a separate investigation.

2. There are equivalent forms of (1) which lead to the same equation (2). Let Φ be any function of ϕ, say

$$\Phi = f(\phi);$$

then, if c be the value of $f(a)$, the equation (1) may be replaced by

$$\Phi = c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)',$$

where Φ is a function of x, y, z, u,... and c is a constant. The same small variations to which the variables are subjected are now connected by the equation

$$\frac{\partial \Phi}{\partial x} dx + \frac{\partial \Phi}{\partial y} dy + \frac{\partial \Phi}{\partial z} dz + \frac{\partial \Phi}{\partial u} du + \ldots = 0.$$

But, since x, y, z, u,... enter into Φ only through ϕ, we have

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Phi}{\partial \phi}\frac{\partial \phi}{\partial x} = \frac{\partial \Phi}{\partial \phi}\mu P = P\mu f'(\phi),$$

$$\frac{\partial \Phi}{\partial y} = \frac{\partial \Phi}{\partial \phi}\frac{\partial \phi}{\partial y} = \frac{\partial \Phi}{\partial \phi}\mu Q = Q\mu f'(\phi),$$

with similar equations; and therefore the equation connecting the variations reduces to (2) as before on the removal of the factor M, where

$$M = \mu f'(\phi);$$

and therefore M is an integrating factor which will enable us to obtain the equation (1)$'$. Hence for every form of f leading to an integral equation new in form, we have a corresponding integrating factor.

It is convenient to call a function of the variables a *solution* of (2), if (2) be satisfied in virtue of the relation obtained by equating that function to a constant; thus ϕ and Φ are solutions of (2). The result just obtained shews that, *if two quantities be functions of one another, they are solutions of the same equation.*

3. Conversely, if the differential equation

$$Pdx + Qdy + Rdz + Sdu + \ldots = 0$$

(assumed to be the only relation connecting the differentials of the variables) can be satisfied in virtue of a single integral equation *all its solutions are equivalent to one another*, that is, one solution is sufficient for the construction of all the solutions. For let

$$\phi = \phi\,(x,\ y,\ z,\ u, \ldots\ldots)$$

$$\Phi = \Phi\,(x,\ y,\ z,\ u, \ldots\ldots)$$

be two solutions, so that we have

$$d\phi = 0, \quad d\Phi = 0.$$

When x is eliminated between the two integral equations, the resulting equation is of the form

$$F\,(\phi,\ \Phi,\ y,\ z,\ u, \ldots\ldots) = 0.$$

Hence

$$\frac{\partial F}{\partial \phi}\,d\phi + \frac{\partial F}{\partial \Phi}\,d\Phi + \frac{\partial F}{\partial y}\,dy + \frac{\partial F}{\partial z}\,dz + \frac{\partial F}{\partial u}\,du + \ldots = 0,$$

so that

$$\frac{\partial F}{\partial y}\,dy + \frac{\partial F}{\partial z}\,dz + \frac{\partial F}{\partial u}\,du + \ldots = 0,$$

a differential equation among the same variables and distinct from the original differential equation because the variation dx does not occur. But as the original equation is the only relation connecting the differentials of the variables, it follows that the new equation, not satisfied in virtue of that original equation, is evanescent; and therefore

$$\frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = 0, \quad \frac{\partial F}{\partial u} = 0, \ldots\ldots$$

so that F is explicitly independent of $y, z, u, \ldots$ and takes the form

$$F\,(\Phi,\ \phi) = 0.$$

Hence Φ can be obtained in terms of ϕ, which proves the proposition.

Loosely worded?

4. Again the integrating factors M and μ corresponding to Φ and ϕ are such that

$$M \div \mu = f'(\phi),$$

dependent on ϕ, x, y, z

and $f'(\phi)$, a function of the solution ϕ, is (§ 2) a solution; hence the *quotient of two integrating factors, if not a constant, is a solution of the equation.*

equation?

And, *if ϕ be the solution determined by the factor μ, every other factor is of the form $\mu . \lambda(\phi)$ where $\lambda(\phi)$ is a function of ϕ.*

5. When a differential equation of the form at present under consideration is given, there is not necessarily a single integral equation in virtue of which it is satisfied. The conditions that this may be so are that the coefficients $P, Q, R, S, \ldots$ of the differentials are proportional to the partial differential coefficients of one function, conditions which are not satisfied for any set of arbitrarily assigned quantities $P, Q, R, S, \ldots$; and these conditions lead to relations between the quantities, which must be satisfied in order that the differential equation may have a single equation as its integral equivalent. We proceed to obtain these relations.

Let the differential equation be

$$X_1 dx_1 + X_2 dx_2 + \ldots\ldots\ldots + X_p dx_p = 0 \ldots\ldots\ldots\ldots (3),$$

and suppose it derived from the equation

$$\phi(x_1, x_2, \ldots\ldots, x_p) = \text{constant} \ldots\ldots\ldots\ldots\ldots (4),$$

by the rejection of a factor μ after differentiation; then we have

$$\mu X_r = \frac{\partial \phi}{\partial x_r} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

for values $1, 2, \ldots, p$ of r. From the equations (5) it follows that

$$\frac{\partial}{\partial x_m}(\mu X_n) = \frac{\partial^2 \phi}{\partial x_m \partial x_n} = \frac{\partial}{\partial x_n}(\mu X_m)$$

for any two indices m and n; and therefore

$$X_n \frac{\partial \mu}{\partial x_m} - X_m \frac{\partial \mu}{\partial x_n} = \mu \left(\frac{\partial X_m}{\partial x_n} - \frac{\partial X_n}{\partial x_m} \right)$$

$$= \mu a_{m,n},$$

where

$$a_{m,n} = \frac{\partial X_m}{\partial x_n} - \frac{\partial X_n}{\partial x_m} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

If r denote any other index, we have similarly

$$X_r \frac{\partial \mu}{\partial x_n} - X_n \frac{\partial \mu}{\partial x_r} = \mu a_{n,r},$$

and

$$X_m \frac{\partial \mu}{\partial x_r} - X_r \frac{\partial \mu}{\partial x_m} = \mu a_{r,m}.$$

Multiplying these three equations by X_r, X_m, X_n respectively and adding, we have

$$0 = \mu (a_{m,n} X_r + a_{n,r} X_m + a_{r,m} X_n),$$

or, since μ does not vanish, identically

$$a_{m,n} X_r + a_{n,r} X_m + a_{r,m} X_n = 0 \quad \text{..............(7)}.$$

This equation, which is evanescent if two of the indices be the same, holds for any combination of three of the indices of the series 1, 2,..., p; and therefore the number of equations, of the same form as (7), between the quantities X is

$$\tfrac{1}{6} p (p-1)(p-2),$$

each being identically satisfied.

6. These equations are not, however, all independent of one another. Taking any other index s, distinct from m, n, r we have, in addition to (7),

$$a_{s,m} X_r + a_{r,s} X_m + a_{m,r} X_s = 0 \quad \text{..............(7)}',$$

$$a_{m,s} X_n + a_{s,n} X_m + a_{n,m} X_s = 0 \quad \text{..............(7)}'',$$

and lastly

$$a_{n,r} X_s + a_{r,s} X_n + a_{s,n} X_r = 0 \quad \text{..............(7)}'''.$$

Multiplying (7), (7)′, (7)″ by X_s, X_n, X_r respectively and adding we have, in virtue of the property

$$a_{k,l} = -a_{l,k}$$

for all pairs of indices, the relation

$$X_m (a_{n,r} X_s + a_{r,s} X_n + a_{s,n} X_r) = 0,$$

which is, in effect, the equation (7)‴ since X_m does not vanish. Hence of the four equations, each involving three of a set of four indices, only three are independent; any one of the four equations can be deduced from the other three.

Let us consider as the three independent equations those which involve m, n, r; m, r, s; m, s, n; in the foregoing set they

are (7), (7)′, (7)″. If between (7)′ and (7)″ we eliminate X_s we have

$$a_{m,n}\, a_{s,m}\, X_r + (a_{m,n}\, a_{r,s} + a_{m,r}\, a_{s,n})\, X_m + a_{r,m}\, a_{s,m}\, X_n = 0\,;$$

to which if (7) multiplied by $a_{m,s}$ be added, we have

$$a_{m,n}\, a_{r,s} + a_{m,r}\, a_{s,n} + a_{m,s}\, a_{n,r} = 0$$

on the rejection of the factor X_m. This last equation is satisfied because (7), (7)′, (7)″ are satisfied; if in any case desirable, it could replace any one of the three.

Since the equation which involves the indices n, r, s is deducible from the three which involve pairs of these indices and some other index the same for the three, we shall obtain all the independent equations by taking some definite index, say 1, and forming all the sets of three. The aggregate of all these sets is really the aggregate obtained by combining the index 1 with every pair of indices other than 1, that is, with every pair formed from $2, 3, \ldots, p$; and the equations in this aggregate are independent of one another. Hence the number of independent equations of condition is

$$\tfrac{1}{2}(p-1)(p-2).$$

It is to be noticed that, if μ be unity, then the equations are all satisfied in virtue of the vanishing of the quantities $a_{m,n}$; and the equations of condition are in this case

$$a_{m,n} = 0,$$

their number being $\frac{1}{2}p(p-1)$. The extra number of conditions arises from the additional supposed limitation that the equation is exact as given and therefore requires no factor to make it so.

7. The conditions of the type (7) are a necessary consequence of the supposition that the differential equation (3) can be made an exact differential; it will now be shewn conversely that, *if the conditions* (7) *be satisfied, then the differential equation can be made exact.*

It is known from the theory of equations which involve only two variables x and y that for an equation

$$Pdx + Qdy = 0$$

there exists a function $\theta(x, y)$ such that the differential equation is satisfied in virtue of the relation

$$\theta(x, y) = \text{constant},$$

and therefore that P and Q are proportional to the derivatives of θ with regard to x and y respectively. Considering then X_1 and X_2 as functions of x_1 and x_2, we infer that there exists a function u of x_1 and x_2 such that for some quantity λ we may write

$$\lambda X_1 = \frac{\partial u}{\partial x_1}, \quad \lambda X_2 = \frac{\partial u}{\partial x_2};$$

and the function u will involve the other quantities which occur in X_1 and X_2, viz. $x_3, x_4, \ldots, x_p$, the presence of which does not however affect derivation with regard to x_1 and x_2. But it may not be inferred that the remaining coefficients in the equation are similarly proportional to the remaining derivatives; and we therefore take

$$\lambda X_r - \frac{\partial u}{\partial x_r} = Y_r \qquad (8)$$

(for $r = 3, 4, \ldots, p$), where Y_r may be considered known when u is, as it is supposed to be, known.

These new quantities Y_r will satisfy certain equations, which are derivable in virtue of the aggregate (7). It has been seen that only three of the four equations which involve four indices need be retained in that aggregate; and, as already (§ 6) explained, the retained equations will be taken to be made up of

(i) the $p-2$ equations involving the indices $1, 2, r$,

(ii) the $\frac{1}{2}(p-2)(p-3)$ .... $1, r, s$,

where r and s are different from one another and are terms in the series $3, 4, \ldots, p$. This set of combinations is evidently the set obtained by combining the index 1 with every pair formed from $2, 3, \ldots, p$.

8. Considering the first of the two series of retained equations we have, for each index r,

$$\frac{\partial}{\partial x_1}(\lambda X_r - Y_r) = \frac{\partial^2 u}{\partial x_r \partial x_1} = \frac{\partial}{\partial x_r}(\lambda X_1),$$

so that
$$X_r \frac{\partial \lambda}{\partial x_1} - X_1 \frac{\partial \lambda}{\partial x_r} = \lambda a_{1,r} + \frac{\partial Y_r}{\partial x_1}.$$

Similarly
$$X_r \frac{\partial \lambda}{\partial x_2} - X_2 \frac{\partial \lambda}{\partial x_r} = \lambda a_{2,r} + \frac{\partial Y_r}{\partial x_2},$$

and
$$X_2 \frac{\partial \lambda}{\partial x_1} - X_1 \frac{\partial \lambda}{\partial x_2} = \lambda a_{1,2}.$$

Now of the aggregate (7) the retained equation which involves the indices 1, 2, r is

$$a_{1,2} X_r + a_{2,r} X_1 + a_{r,1} X_2 = 0;$$

so that, multiplying the preceding equations by $-X_2$, X_1, X_r respectively, adding and using the condition-relation, we have

$$X_1 \frac{\partial Y_r}{\partial x_2} - X_2 \frac{\partial Y_r}{\partial x_1} = 0.$$

This is the only equation of series (i) which involves Y_r alone; all the equations of series (ii) involve two of the quantities Y, and the import of such equations will be indicated immediately. We may thus regard the preceding equation as an equation determining the form of Y_r. It is a linear partial differential equation of the first order; to obtain the most general solution we construct $p-1$ independent integrals of the subsidiary equations

$$\frac{dx_1}{-X_2} = \frac{dx_2}{X_1} = \frac{dx_3}{0} = \frac{dx_4}{0} = \ldots\ldots = \frac{dx_p}{0}.$$

There are $p-2$ integrals at once given in the form

$$x_r = \text{constant} \quad (r = 3, 4, \ldots, p);$$

so that only one more is needed, to be given by

$$X_1 dx_1 + X_2 dx_2 = 0,$$

or $$\lambda \left(\frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 \right) = 0,$$

i.e., $$\frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 = 0.$$

But in the simultaneous system dx_3, dx_4,... all vanish, and therefore the last equation may be taken in the form

$$\frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \frac{\partial u}{\partial x_3} dx_3 + \ldots\ldots = 0$$

viz., $$du = 0;$$

and therefore the other required integral is

$$u = \text{constant}.$$

Hence, by the theory of linear partial differential equations, it

follows that Y_r is of the form

$$Y_r = f_r(u, x_3, x_4, \ldots, x_p) \text{ .... } (9),$$

where f_r may at present imply *any* function of the arguments.

Multiplying the equation (3) by λ and substituting from (8) for the quantities λX, the new form of the equation is

$$du + Y_3 dx_3 + Y_4 dx_4 + \ldots\ldots + Y_p dx_p = 0 \text{ .... } (3)',$$

where the quantities Y_r are given by the equations (9). Hence, in virtue of the first series of retained equations of § 7, *the given differential equation* (3) *has been transformed into another* (3)′ *involving one variable fewer.*

9. Consider now the second of the two series of retained equations of § 7. Taking a typical equation of the series, we have

$$a_{1,r} X_s + a_{r,s} X_1 + a_{s,1} X_r = 0.$$

But
$$\frac{\partial}{\partial x_r}(\lambda X_s - Y_s) = \frac{\partial^2 u}{\partial x_r \partial x_s} = \frac{\partial}{\partial x_s}(\lambda X_r - Y_r),$$

so that
$$X_s \frac{\partial \lambda}{\partial x_r} - X_r \frac{\partial \lambda}{\partial x_s} = \lambda a_{r,s} + \frac{\partial Y_s}{\partial x_r} - \frac{\partial Y_r}{\partial x_s}.$$

And, before, we had

$$X_r \frac{\partial \lambda}{\partial x_1} - X_1 \frac{\partial \lambda}{\partial x_r} = \lambda a_{1,r} + \frac{\partial Y_r}{\partial x_1};$$

similarly
$$X_s \frac{\partial \lambda}{\partial x_1} - X_1 \frac{\partial \lambda}{\partial x_s} = \lambda a_{1,s} + \frac{\partial Y_s}{\partial x_1}.$$

Multiplying these three equations by X_1, X_s, $-X_r$ respectively, adding and using the former relation, we have

$$X_1\left(\frac{\partial Y_s}{\partial x_r} - \frac{\partial Y_r}{\partial x_s}\right) + X_s \frac{\partial Y_r}{\partial x_1} - X_r \frac{\partial Y_s}{\partial x_1} = 0.$$

In this equation the quantities Y_r and Y_s are functions of $x_1, x_2, \ldots, x_p$ as given by (8). When we take their forms as determined in (9), we have

$$\frac{\partial Y_r}{\partial x_1} = \frac{\partial f_r}{\partial u}\frac{\partial u}{\partial x_1} = \lambda X_1 \frac{\partial f_r}{\partial u},$$

$$\frac{\partial Y_r}{\partial x_s} = \frac{\partial f_r}{\partial u}\frac{\partial u}{\partial x_s} + \frac{\partial f_r}{\partial x_s} = (\lambda X_s - f_s)\frac{\partial f_r}{\partial u} + \frac{\partial f_r}{\partial x_s},$$

and therefore

$$X_1 \frac{\partial Y_r}{\partial x_s} - X_s \frac{\partial Y_r}{\partial x_1} = X_1 \left(\frac{\partial f_r}{\partial x_s} - f_s \frac{\partial f_r}{\partial u} \right).$$

Similarly $$X_1 \frac{\partial Y_s}{\partial x_r} - X_r \frac{\partial Y_s}{\partial x_1} = X_1 \left(\frac{\partial f_s}{\partial x_r} - f_r \frac{\partial f_s}{\partial u} \right);$$

and therefore the above equation becomes, on the rejection of the non-vanishing factor $-X_1$,

$$\frac{\partial f_r}{\partial x_s} - \frac{\partial f_s}{\partial x_r} + f_r \frac{\partial f_s}{\partial u} - f_s \frac{\partial f_r}{\partial u} = 0 \quad \text{...............(10)},$$

for all the combinations of the indices r and s. These are the equations derived from the second series of retained equations of § 7.

It thus follows that the coefficients of the transformed equation

$$du + f_3 dx_3 + f_4 dx_4 + \ldots\ldots + f_p dx_p = 0 \quad \text{...........(3)}',$$

equivalent to (3), are subject to the conditions (10).

10. Since the new equation (3)′ is equivalent to (3), let us find the set of conditions which bear the same relation to (3)′ as the set (7) bear to (3). Associating a subscript index 0 with u and defining $b_{r,s}$ by the equation

$$b_{r,s} = \frac{\partial f_r}{\partial x_s} - \frac{\partial f_s}{\partial x_r}$$

for all pairs of the indices 0, 3, 4, ..., p, the complete set of conditions, associated with (3)′ and similar to (7), are

$$b_{r,s} f_q + b_{s,q} f_r + b_{q,r} f_s = 0,$$

in number equal to $\frac{1}{6}(p-1)(p-2)(p-3)$. But of this number only $\frac{1}{2}(p-2)(p-3)$ are independent; and an independent set, as in the earlier case, can be obtained by retaining all those equations which involve any one index, say 0, with all possible pairs of indices from 3, 4, ..., p. Taking then $q = 0$ we have

$$f_q = 1,$$

$$b_{s,q} = \frac{\partial f_s}{\partial u},$$

$$b_{q,r} = -\frac{\partial f_r}{\partial u};$$

and thus the above condition is,

$$\frac{\partial f_r}{\partial x_s} - \frac{\partial f_s}{\partial x_r} + f_r \frac{\partial f_s}{\partial u} - f_s \frac{\partial f_r}{\partial u} = 0,$$

which is the typical equation of the set (10). Since the possible combinations of indices are the same for the two sets it follows that the *equations of condition, constituted by the system* (10), *have the same relation to the transformed differential equation as the equations of condition, constituted by the system* (7), *have to the original differential equation.*

11. If, then, a differential equation containing $p-1$ variables, such that the associated system of conditions among the coefficients is satisfied, can be represented by a single integral equation, it follows that a differential equation containing p variables, such that the associated system of conditions among its coefficients is satisfied, can also be represented by a single integral equation. For the preceding investigation shews that the equation (3), subject to the conditions (7), can be reduced to the equation (3)′ subject to the conditions (10); so that, if an integral equation equivalent to the latter be

$$F\{u,\ x_3,\ x_4,\ \ldots\ldots,\ x_p\} = \text{constant},$$

then an integral equation equivalent to the former is

$$F\{f(x_1,\ x_2,\ x_3,\ \ldots\ldots,\ x_p),\ x_3,\ x_4,\ \ldots\ldots,\ x_p\} = \text{constant},$$

where f is a function known to exist.

We therefore use the method of induction. In the case when $p=3$ the equation is

$$X_1 dx_1 + X_2 dx_2 + X_3 dx_3 = 0$$

and the single equation of condition is

$$a_{2,3} X_1 + a_{3,1} X_2 + a_{1,2} X_3 = 0,$$

the preceding investigation shews that a function u exists, such that the equation can be transformed to

$$du + \phi(u,\ x_3)\, dx_3 = 0,$$

where ϕ is subject to no condition. But it is known from the theory of differential equations in two variables that some function F exists such that, in virtue of the equation

$$F(u,\ x_3) = \text{constant},$$

the transformed equation is satisfied. Hence the untransformed equation in three variables, having its coefficients subject to a single condition, can be satisfied in virtue of a single integral equation

$$F\{f(x_1, x_2, x_3), x_3\} = \text{constant}.$$

Hence the method of induction leads us to infer that the equation (3), having its coefficients subject to the conditions (7), can be satisfied by means of a single integral equation.

The result of the investigation may be enunciated as follows:—

If the differential equation

$$X_1 dx_1 + X_2 dx_2 + \ldots\ldots + X_p dx_p = 0$$

can be satisfied by means of a single integral equation of the form

$$\phi(x_1, x_2, \ldots\ldots, x_p) = \textit{constant},$$

then the system of conditions

$$a_{m,n} X_r + a_{n,r} X_m + a_{r,m} X_n = 0$$

$\left(\textit{where } a_{m,n} = \dfrac{\partial X_m}{\partial x_n} - \dfrac{\partial X_n}{\partial x_m}\right)$ *is satisfied identically for all combinations of the indices m, n, r from the series* 1, 2, 3, ..., *p; and of these only* $\frac{1}{2}(p-1)(p-2)$ *are independent. Conversely, if the system of conditions be satisfied identically for all combinations of the indices, then the differential equation can be satisfied by means of a single integral equation of the form*

$$\phi(x_1, x_2, \ldots\ldots, x_p) = \textit{constant}.$$

The following corollary can be inferred from the preceding analysis:—

Let $X_1, X_2, \ldots, X_p$ denote functions of independent variables $x_1, x_2, \ldots, x_p$. It is known that, whatever be the quantities X, there exists some function u_1 such that

$$X_1 : X_2 = \frac{\partial u_1}{\partial x_1} : \frac{\partial u_1}{\partial x_2}.$$

If a relation

$$a_{23} X_1 + a_{31} X_2 + a_{12} X_3 = 0$$

be satisfied, then there exists some function u_2 of the variables such that

$$X_1 : X_2 : X_3 = \frac{\partial u_2}{\partial x_1} : \frac{\partial u_2}{\partial x_2} : \frac{\partial u_2}{\partial x_3}.$$

If relations $$a_{rs}X_1 + a_{s1}X_r + a_{1r}X_s = 0$$
(for r, $s = 2, 3, 4$) be satisfied, then there exists some function u_3 of the variables such that

$$X_1 : X_2 : X_3 : X_4 = \frac{\partial u_3}{\partial x_1} : \frac{\partial u_3}{\partial x_2} : \frac{\partial u_3}{\partial x_3} : \frac{\partial u_3}{\partial x_4}.$$

And so on.

12. Let it now be supposed that the differential equation can be satisfied by a single integral equation; we proceed to obtain that equation.

One method of derivation of the solution would be the carrying out of the successive reductions indicated in the investigation just completed: it is practically Euler's method. It may be shortly summarised as follows:—

First, let all the variables except x_1 and x_2 be considered constant; and on this hypothesis let a solution of

$$X_1 dx_1 + X_2 dx_2 = 0$$

be
$$u = u(x_1, x_2, \ldots) = \text{constant},$$

so that
$$X_1 : X_2 = \frac{\partial u}{\partial x_1} : \frac{\partial u}{\partial x_2}.$$

By means of the new variable u, where

$$u = u(x_1, x_2, x_3, \ldots\ldots, x_p),$$

eliminate from the differential equation

$$X_1 dx_1 + X_2 dx_2 + X_3 dx_3 + \ldots\ldots + X_p dx_p = 0$$

the quantities x_1, x_2, dx_1, dx_2—an elimination which will be found possible when the equations of condition are satisfied—so that the equation takes the form

$$du + Y_3 dx_3 + Y_4 dx_4 + \ldots\ldots + Y_p dx_p = 0,$$

in which Y_3, Y_4, ..., Y_p are now functions of u, x_3, x_4, ... x_p.

Secondly, let the same process be repeated with u and x_3 as the variables initially considered; then, if

$$v = v(u, x_3, x_4, \ldots\ldots, x_p)$$

be the new variable, the equation similarly comes to be

$$dv + Z_4 dx_4 + \ldots\ldots + Z_p dx_p = 0;$$

and so on for the various steps in succession, which in number cannot be greater than $p - 1$. If at any stage the transformed

equation should have all its coefficients free from any one variable, it can be made an exact equation on division throughout by those factors of the coefficient of the variation of that variable which do not involve it: so that a single integration will then lead to the part of the desired integral which depends upon that variable.

13. But we may proceed as follows. It is known that the expression

$$X_1 dx_1 + X_2 dx_2 + \ldots\ldots + X_p dx_p$$

becomes, after multiplication by some factor μ, an exact differential $d\phi$; so that if this factor can be obtained the required solution will be obtained after the single integration required to evaluate

$$\phi = \int \mu \left(X_1 dx_1 + X_2 dx_2 + \ldots\ldots + X_p dx_p\right) = \text{constant}.$$

We have $$\mu X_r = \frac{\partial \phi}{\partial x_r},$$

so that taking the complete variation of both sides we have

$$d\mu \,.\, X_r + \mu \sum_{s=1}^{p} \frac{\partial X_r}{\partial x_s}\, dx_s = \sum_{s=1}^{p} \frac{\partial^2 \phi}{\partial x_r \partial x_s}\, dx_s.$$

But on the right-hand side

$$\frac{\partial^2 \phi}{\partial x_r \partial x_s} = \mu \frac{\partial X_s}{\partial x_r} + X_s \frac{\partial \mu}{\partial x_r}$$

for all indices s; and therefore

$$d\mu \,.\, X_r + \mu \sum_{s=1}^{p} \frac{\partial X_r}{\partial x_s}\, dx_s = \mu \sum_{s=1}^{p} \frac{\partial X_s}{\partial x_r}\, dx_s + \frac{\partial \mu}{\partial x_r} \sum_{s=1}^{p} X_s dx_s$$

or $$\frac{d\mu}{\mu} X_r + \sum_{s=1}^{p} a_{r,s} dx_s = \frac{1}{\mu^2} \frac{\partial \mu}{\partial x_r}\, d\phi.$$

But the complete variation of ϕ is zero, when the variables are connected as by the given differential equation; so that the foregoing equation becomes

$$-\frac{d\mu}{\mu} = \frac{1}{X_r} \left\{a_{r,1} dx_1 + a_{r,2} dx_2 + \ldots\ldots + a_{r,p} dx_p\right\} \ldots\ldots (11)$$

the term in dx_r being absent from the expression on the right-hand side. This equation is valid in virtue of the relation $d\phi = 0$ or, what is the same thing, in virtue of the given differential equation; the left-hand side is a perfect differential, and therefore the right-hand side also is a perfect differential subject to the relation $d\phi = 0$.

Since r may be any one of the quantities 1, 2, 3, ..., p, we have the set of equations

$$\left.\begin{aligned}
-\frac{d\mu}{\mu} &= \frac{1}{X_1}\{0\,.\,dx_1 + a_{12}dx_2 + a_{13}dx_3 + \ldots\ldots + a_{1p}dx_p\} \\
&= \frac{1}{X_2}\{a_{21}dx_1 + 0\,.\,dx_2 + a_{23}dx_3 + \ldots\ldots + a_{2p}dx_p\} \\
&= \frac{1}{X_3}\{a_{31}dx_1 + a_{32}dx_2 + 0\,.\,dx_3 + \ldots\ldots + a_{3p}dx_p\} \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&= \frac{1}{X_p}\{a_{p1}dx_1 + a_{p2}dx_2 + a_{p3}dx_3 + \ldots\ldots + 0\,.\,dx_p\}
\end{aligned}\right\} \ldots(12).$$

It is easy to verify that all the expressions for $-\dfrac{d\mu}{\mu}$ in this system are equivalent to one another, on account of the differential equation itself and the conditions (7) which are satisfied by the coefficients in that equation.

14. Though the fractions in (12) are equal to one another and to $-d\log\mu$ in virtue of the differential equation, it often happens that no one of the fractions is an exact differential in the form there given. If any one of them be an exact differential in the form in which it occurs, the value of μ is immediately derivable. Thus if all the quantities a in the numerator of any fraction vanish—if for instance X_1 be a function of x_1 alone and no other coefficient X involve x_1—then we may take $\mu = 1$; and the original equation is an exact equation.

Again, any value of μ which satisfies these equations will prove sufficient for our purpose; and the simpler that value is, the easier in general will be the subsequent integration for ϕ. Now from (12) we have

$$-\frac{d\mu}{\mu} = \frac{\sum\limits_{r=1}^{p}\sum\limits_{s=1}^{p} Y_r a_{rs} dx_s}{\sum\limits_{r=1}^{p} Y_r X_r},$$

whatever the quantities Y_r may be; and it often happens in practice that this combination of the equal fractions leads to a perfect differential by means of suitably chosen quantities Y. Such is especially the case when X_r is a function of x_r and also is symmetric in all the other variables.

Further, the quotient of two values of μ is a solution of the differential equation (§ 4). For if μ and μ' be those values we have

$$\frac{d\mu}{\mu}=\frac{d\mu'}{\mu'}$$

or μ'/μ is constant, a result obtained on the introduction of the condition $d\phi=0$ or $\phi=$ constant. Hence μ'/μ and ϕ are constant together; and therefore* there is some functional relation between them, which may be represented in the form $\mu'/\mu=f(\phi)$. But since ϕ is a solution so also (§ 3) is $f(\phi)$; and therefore μ'/μ is a solution of the original equation.

If then two values of μ can be obtained from (12) and their quotient be not constant, a solution (and therefore all solutions) can be given by equating that quotient to a constant.

We can, from this point of view, deduce the earlier inference (§ 4) as to the general form of μ, and also the result that, if three values of μ can be obtained from (12), there is a homogeneous relation among them.

15. *Ex.* 1. The requisite conditions are all satisfied for the equation

$$z(y+z)\,dx+z(u-x)\,dy+y(x-u)\,dz+y(y+z)\,du=0;$$

for

$$a_{12}=\ 2z,\quad a_{13}=\ 2z,\quad a_{14}=0.$$
$$a_{34}=-2y,\quad a_{24}=-2y,\quad a_{23}=2(u-x).$$

Taking the first of the fractions for $-d\mu/\mu$ we have

$$-\frac{d\mu}{\mu}=\frac{2zdy+2zdz}{z(y+z)}=2\frac{dy+dz}{y+z},$$

so that

$$\mu=(y+z)^{-2},$$

the same value being given by the fourth of the fractions, and also by a combination of the second and third. Then

$$(y+z)^{-2}\,\{z(y+z)\,dx+z(u-x)\,dy+y(x-u)\,dz+y(y+z)\,du\}$$

is an exact differential; and a solution of the original equation is given by

$$\frac{xz+uy}{y+z}=\text{constant}.$$

* For since μ'/μ and ϕ are functions of the variables we have, after elimination of one of the variables, say x_1, from the two equations expressing those functions, a result of the form

$$\frac{\mu'}{\mu}=f(\phi,\,x_2,\,\ldots,\,x_p).$$

Since $\phi=$constant implies $\mu'/\mu=$constant, the integral leads to a new relation between $x_2,\,\ldots,\,x_p$ which is different from the former one. As this cannot exist, it must be evanescent as a relation among the variables; and therefore we have the result in the text.

Ex. 2. The requisite conditions are similarly all satisfied for the equation

$$(y+z)(z+u)(u+y)\,dx+\text{three similar terms}=0.$$

Taking the first and the second of the four fractions, we have

$$-\frac{d\mu}{\mu}=\frac{2(z+u)(y-x)\,dy+2(y+u)(z-x)\,dz+2(y+z)(u-x)\,du}{(y+z)(z+u)(u+y)}$$

$$=\frac{2(z+u)(x-y)\,dx+2(x+u)(z-y)\,dz+2(x+z)(u-y)\,du}{(z+x)(x+u)(u+z)}$$

$$=\frac{\text{second numerator}-\text{first numerator}}{\text{second denominator}-\text{first denominator}}.$$

Removing from the numerator and the denominator of this fraction the common factor $(u+z)(x-y)$, we have

$$-\frac{d\mu}{\mu}=2\frac{dx+dy+dz+du}{x+y+z+u},$$

so that

$$\mu=(x+y+z+u)^{-2}.$$

The solution is now easily found to be

$$\frac{xyz+yzu+zux+uxy}{x+y+z+u}=\text{constant}.$$

The derivation of the solution by the method of § 12 is rather long.

Ex. 3. In the case of an equation in three variables such as

$$Pdx+Qdy+Rdz=0,$$

we have

$$a_{23}=\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}=X,\quad a_{31}=\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}=Y,\quad a_{12}=\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}=Z;$$

and the condition of integrability is

$$PX+QY+RZ=0.$$

The equations for the factor are

$$-\frac{d\mu}{\mu}=\frac{Zdy-Ydz}{P}=\frac{Xdz-Zdx}{Q}=\frac{Ydx-Xdy}{R}$$

$$=\frac{\begin{vmatrix} dx, & dy, & dz \\ X, & Y, & Z \\ a, & b, & c \end{vmatrix}}{aP+bQ+cR},$$

where a, b, c are any quantities whatever.

As a special case let

$$P=x^2y-y^3-y^2z,\quad Q=xy^2-x^3-x^2z,\quad R=xy^2+x^2y;$$

then $\quad X=-2x(x+y),\quad Y=2y(x+y),\quad Z=2(x-y)(2x+2y+z).$

From the third of the fractions it follows that

$$-\frac{d\mu}{\mu}=\frac{(2ydx+2xdy)(x+y)}{xy^2+x^2y}=2\frac{ydx+xdy}{xy},$$

so that

$$\mu=x^{-2}y^{-2}.$$

Again, taking

$$-\frac{d\mu'}{\mu'}=\frac{Zdy-Ydz-(Xdz-Zdx)}{P-Q}$$

$$=\frac{Z(dy+dx)-(X+Y)\,dz}{P-Q}$$

$$=\frac{2(x-y)\{(2x+2y+z)(dx+dy)+(x+y)\,dz\}}{(x-y)(x+y)(x+y+z)},$$

we have the new fraction a perfect differential on the removal of the factor $x-y$; so that

$$\mu'=(x+y)^{-2}(x+y+z)^{-2}.$$

Since we now have two integrating factors which do not necessarily bear a constant ratio, a solution of the original equation is

$$\frac{\mu}{\mu'}=\text{constant},$$

or, extracting the square root,

$$\frac{(x+y)(x+y+z)}{xy}=\text{constant},$$

the required solution.

16. For the equation in three variables Bertrand* adopts the following method. In the notation of § 15, he constructs the subsidiary equations

$$\frac{dx}{X}=\frac{dy}{Y}=\frac{dz}{Z},$$

and obtains two independent integrals of these, say

$$\alpha=\phi_1(x, y, z), \quad \beta=\phi_2(x, y, z);$$

so that

$$X\frac{\partial\phi_1}{\partial x}+Y\frac{\partial\phi_1}{\partial y}+Z\frac{\partial\phi_1}{\partial z}=0,$$

$$X\frac{\partial\phi_2}{\partial x}+Y\frac{\partial\phi_2}{\partial y}+Z\frac{\partial\phi_2}{\partial z}=0.$$

But since

$$\mu P=\frac{\partial\phi}{\partial x}, \quad \mu Q=\frac{\partial\phi}{\partial y}, \quad \mu R=\frac{\partial\phi}{\partial z},$$

(where $\phi=$ constant is the solution of the equation), the condition of integrability gives

$$X\frac{\partial\phi}{\partial x}+Y\frac{\partial\phi}{\partial y}+Z\frac{\partial\phi}{\partial z}=0.$$

* *Comptes Rendus*, t. lxxxiii. (1876), pp. 1191—1195.

Eliminating X, Y, Z we have

$$\frac{\partial(\phi, \phi_1, \phi_2)}{\partial(x, y, z)} = 0,$$

and therefore ϕ can be expressed as a function of ϕ_1 and ϕ_2; hence some function f exists such that the solution of the equation is

$$f(\phi_1, \phi_2) = \phi = \text{constant},$$

say
$$f(\alpha, \beta) = \text{constant}.$$

Hence the equation can be transformed to

$$Md\alpha + Nd\beta = 0,$$

where $M : N = \dfrac{\partial f}{\partial \alpha} : \dfrac{\partial f}{\partial \beta}$, when α and β are taken as variables; and any solution of the last equation is also a solution of the original equation.

It will be noticed that the subsidiary equations are evanescent when the given equation is exact.

Ex. For the special example given in Ex. 3 of § 15 the subsidiary system is, on dropping a factor 2,

$$\frac{-dx}{x(x+y)} = \frac{dy}{y(x+y)} = \frac{dz}{(x-y)(2x+2y+z)}.$$

One integral is given by the first two fractions in the form

$$xy = \alpha.$$

From the fractions we construct the relation

$$\frac{dx+dy}{x+y} = -\frac{dx+dy+dz}{x+y+z}$$

on dropping a factor $x-y$; so that a second (independent) integral is given by

$$(x+y)(x+y+z) = \beta.$$

Since the equation is capable of transformation to the form

$$Md\alpha + Nd\beta = 0,$$

we have, on substituting for α and β, the equations

$$My + N(2x+2y+z) = y(x^2-y^2-yz),$$
$$Mx + N(2x+2y+z) = x(y^2-x^2-xz),$$
$$N(x+y) = xy(x+y),$$

which are all satisfied by

$$N = xy = \alpha,$$
$$-M = (x+y)(x+y+z) = \beta.$$

The transformed equation is thus

$$\alpha d\beta - \beta d\alpha = 0,$$

having for its integral

$$\frac{\beta}{\alpha} = \text{constant},$$

which, on substitution for α and β, is the same as that before obtained.

Again, since $\mu Q = \dfrac{\partial \phi}{\partial y}$, $\mu R = \dfrac{\partial \phi}{\partial z}$, we have

$$-\mu^2 X = \frac{\partial \mu}{\partial z}\frac{\partial \phi}{\partial y} - \frac{\partial \mu}{\partial y}\frac{\partial \phi}{\partial z},$$

and similarly for the others; so that the subsidiary equations may be taken in the form

$$\frac{dx}{\dfrac{\partial(\phi, \mu)}{\partial(y, z)}} = \frac{dy}{\dfrac{\partial(\phi, \mu)}{\partial(z, x)}} = \frac{dz}{\dfrac{\partial(\phi, \mu)}{\partial(x, y)}}.$$

These are satisfied by

$$\mu = \text{constant},$$
$$\phi = \text{constant};$$

and the constants are expressible in terms of α and β, but are not necessarily equal to them. It thus does not follow that, when two independent solutions of Bertrand's subsidiary system have been obtained, either of them may be taken as an integrating factor of the original equation; in fact, a comparison of the two methods shews that, in the particular example considered,

$$\mu = \alpha^2,$$
$$\mu' = \beta^2,$$

which are the two independent integrating factors*.

17. Bertrand's method, when considered from a different point of view, admits of generalisation to the case of an equation involving more than three variables.

The two new variables α and β, introduced to transform the

* See De Morgan, "On the integrating factor of $Pdx + Qdy + Rdz$," *Quart. Journ*, vol. ii. (1858), pp. 323—326, where he shews that the integrating factor satisfies the equation

$$X\frac{\partial \mu}{\partial x} + Y\frac{\partial \mu}{\partial y} + Z\frac{\partial \mu}{\partial z} = 0,$$

easily obtained by multiplying the three equations by $\dfrac{\partial \mu}{\partial x}, \dfrac{\partial \mu}{\partial y}, \dfrac{\partial \mu}{\partial z}$ respectively and adding.

given equation into one which involves only two variables, are independent integrals of the partial differential equation

$$a_{23}\frac{\partial\theta}{\partial x}+a_{31}\frac{\partial\theta}{\partial y}+a_{12}\frac{\partial\theta}{\partial z}=0;$$

and it is because the required solution ϕ is also an integral of this equation that ϕ can be expressed as some function of α and β thus rendering the transformation possible.

For the purpose of indicating the generalisation, let us consider the equation

$$X_1dx_1+X_2dx_2+X_3dx_3+X_4dx_4=0,$$

the coefficients of which are to be supposed subject to the four conditions of the form

$$a_{23}X_1+a_{31}X_2+a_{12}X_3=0.$$

The quantities a_{mn} are such as to satisfy the relation

$$a_{14}a_{23}+a_{13}a_{42}+a_{12}a_{34}=0,$$

as well as the relations

$$\frac{\partial a_{nr}}{\partial x_m}+\frac{\partial a_{rm}}{\partial x_n}+\frac{\partial a_{mn}}{\partial x_r}=0,$$

the latter set of which admit of immediate verification.

Let if possible some quantity θ satisfy the two equations

$$\left.\begin{aligned}0&=a_{23}\frac{\partial\theta}{\partial x_1}+a_{31}\frac{\partial\theta}{\partial x_2}+a_{12}\frac{\partial\theta}{\partial x_3}\\0&=a_{34}\frac{\partial\theta}{\partial x_1}+a_{41}\frac{\partial\theta}{\partial x_2}\qquad\qquad+a_{13}\frac{\partial\theta}{\partial x_4}\end{aligned}\right\}\text{.........(A)}$$

and therefore also, on account of the relation among the quantities a, the two other equations

$$\left.\begin{aligned}0&=\qquad\qquad a_{34}\frac{\partial\theta}{\partial x_2}+a_{42}\frac{\partial\theta}{\partial x_3}+a_{23}\frac{\partial\theta}{\partial x_4}\\0&=a_{34}\frac{\partial\theta}{\partial x_1}\qquad\qquad+a_{41}\frac{\partial\theta}{\partial x_3}+a_{13}\frac{\partial\theta}{\partial x_4}\end{aligned}\right\}\text{.........(A)}',$$

these two (A)′ being linearly dependent on the former two (A). Now by the ordinary Jacobian theory any quantity θ which satisfies the two equations (A) must also satisfy

$$\left(a_{23}\frac{\partial}{\partial x_1}+a_{31}\frac{\partial}{\partial x_2}+a_{12}\frac{\partial}{\partial x_3}\right)\left(a_{34}\frac{\partial}{\partial x_1}+a_{41}\frac{\partial}{\partial x_2}+a_{13}\frac{\partial}{\partial x_4}\right)\theta$$
$$-\left(a_{34}\frac{\partial}{\partial x_1}+a_{41}\frac{\partial}{\partial x_2}+a_{13}\frac{\partial}{\partial x_4}\right)\left(a_{23}\frac{\partial}{\partial x_1}+a_{31}\frac{\partial}{\partial x_2}+a_{12}\frac{\partial}{\partial x_3}\right)\theta=0,$$

the condition of coexistence. This equation, which must be satisfied by θ, is not new in form; it is easily shewn to be identically satisfied in virtue of the equations (A) and (A)′ and of the relations among the quantities $a_{m,n}$. Hence it follows that the two equations (A), evidently independent of one another, are the only two linearly independent equations to be satisfied by θ; and thus they form a complete Jacobian system*.

Since then there are four variables, of which θ is to be a function, and there are two equations in the complete Jacobian system, it follows (*post*, § 38) that there are two $(=4-2)$ independent simultaneous solutions of the equations in that system. Let α and β be two such solutions; they each satisfy the four equations (A) and (A)′ when they are substituted for θ; and every solution can be expressed in terms of α and β alone.

But we have relations

$$0 = a_{m,n} X_r + a_{n,r} X_m + a_{r,m} X_n,$$

and, if the integral equation equivalent to the differential equation be

$$\psi = \text{constant},$$

we have for every index

$$\mu X_s = \frac{\partial \psi}{\partial x_s};$$

so that substituting and multiplying by μ we have

$$0 = a_{m,n} \frac{\partial \psi}{\partial x_r} + a_{n,r} \frac{\partial \psi}{\partial x_m} + a_{r,m} \frac{\partial \psi}{\partial x_n}$$

for all combinations of the indices. Whence it follows that ψ also is a solution of the four equations (A) and (A)′; and therefore some function f must exist such that

$$\psi = f(\alpha, \beta).$$

The integral equation equivalent to the differential equation thus becomes

$$f(\alpha, \beta) = \text{constant},$$

so that the differential equation may be taken in the form

$$M d\alpha + N d\beta = 0,$$

where $M : N = \dfrac{\partial f}{\partial \alpha} : \dfrac{\partial f}{\partial \beta}$. If then the quantities α and β be taken

* See *Treatise*, § 226; and *post*, §§ 38 *et seq*.

as new variables, the differential equation can be transformed to

$$Md\alpha + Nd\beta = 0,$$

where M and N are functions of α and β alone; the integral of the new form provides the required solution. Hence we have the theorem:—

If the differential equation

$$X_1 dx_1 + X_2 dx_2 + X_3 dx_3 + X_4 dx_4 = 0$$

satisfy all the conditions that it should have a single integral equivalent, and if two independent integrals of the equations

$$\left.\begin{aligned} 0 &= a_{23}\frac{\partial\theta}{\partial x_1} + a_{31}\frac{\partial\theta}{\partial x_2} + a_{12}\frac{\partial\theta}{\partial x_3} \\ 0 &= a_{24}\frac{\partial\theta}{\partial x_1} + a_{41}\frac{\partial\theta}{\partial x_2} + a_{12}\frac{\partial\theta}{\partial x_4} \end{aligned}\right\}*$$

be $\qquad \alpha = \alpha(x_1, x_2, x_3, x_4), \quad \beta = \beta(x_1, x_2, x_3, x_4),$

then when α and β are used as independent variables the equation admits of expression in the form

$$Md\alpha + Nd\beta = 0,$$

where M and N are functions of α and β alone; and the integral of the new equation leads to the solution of the original equation.

And the corresponding theorem for an equation in p variables is:—

If the differential equation

$$X_1 dx_1 + X_2 dx_2 + \ldots\ldots + X_p dx_p = 0$$

satisfy all the conditions that it should have a single integral equivalent, and if two independent integrals of the equations

$$0 = a_{2,m}\frac{\partial\theta}{\partial x_1} + a_{m,1}\frac{\partial\theta}{\partial x_2} + a_{1,2}\frac{\partial\theta}{\partial x_m}$$

(where $m = 3, 4, \ldots, p$) be

$$\alpha = \alpha(x_1, x_2, x_3, \ldots\ldots, x_p),$$
$$\beta = \beta(x_1, x_2, x_3, \ldots\ldots, x_p),$$

* Any two of the four equations (A) and (A)′ may be used as the equations constituting the complete system.

then, when α and β are used as independent variables, the equation admits of expression in the form

$$Md\alpha + Nd\beta = 0,$$

where M and N are functions of α and β alone; and the integral of the new equation leads to the solution of the original equation.

Ex. As an example consider Ex. 1, § 15 replacing x, y, z, u by x_1, x_2, x_3, x_4. The two equations which serve to determine α and β are

$$0 = x_3\frac{\partial\theta}{\partial x_4} - x_2\frac{\partial\theta}{\partial x_1},$$

$$0 = (x_4 - x_1)\frac{\partial\theta}{\partial x_1} - x_3\frac{\partial\theta}{\partial x_2} + x_3\frac{\partial\theta}{\partial x_3} = \Delta\theta,$$

say. The equations subsidiary to the integration of the first are

$$\frac{dx_1}{-x_2} = \frac{dx_2}{0} = \frac{dx_3}{0} = \frac{dx_4}{x_3},$$

three independent integrals of which may be taken in the form

$$\theta_1 = x_2,$$
$$\theta_2 = x_3,$$
$$\theta_3 = x_2x_4 + x_1x_3.$$

Every solution of the first equation can be expressed in terms of these, say,

$$\theta = F(\theta_1, \theta_2, \theta_3);$$

and we have now to determine such forms of F as will satisfy the second equation. Now

$$\Delta\theta_1 = -\theta_2, \quad \Delta\theta_2 = \theta_2, \quad \Delta\theta_3 = 0,$$

so that

$$0 = \frac{\partial F}{\partial\theta_1}\Delta\theta_1 + \frac{\partial F}{\partial\theta_2}\Delta\theta_2 + \frac{\partial F}{\partial\theta_3}\Delta\theta_3$$
$$= -\theta_2\frac{\partial F}{\partial\theta_1} + \theta_2\frac{\partial F}{\partial\theta_2},$$

or removing the factor θ_2

$$0 = -\frac{\partial F}{\partial\theta_1} + \frac{\partial F}{\partial\theta_2}.$$

The equations subsidiary to the integration of this equation are

$$\frac{d\theta_1}{-1} = \frac{d\theta_2}{1} = \frac{d\theta_3}{0},$$

the two necessary integrals of which are

$$\alpha = \theta_1 + \theta_2,$$
$$\beta = \theta_3.$$

Whence, by the theory of such equations, every simultaneous solution of the equations determining θ can be expressed in terms of

$$\alpha=\theta_1+\theta_2=x_2+x_3,$$
$$\beta=\theta_3 \qquad =x_2x_4+x_1x_3.$$

If now to complete the integration we take

$$Md\alpha+Nd\beta=x_3(x_2+x_3)\,dx_1+x_3(x_4-x_1)\,dx_2+x_2(x_1-x_4)\,dx_3+x_2(x_2+x_3)\,dx_4,$$

we have

$$Nx_3=x_3(x_2+x_3),$$
$$M+Nx_4=x_3(x_4-x_1),$$
$$M+Nx_1=x_2(x_1-x_4),$$
$$Nx_2=x_2(x_2+x_3),$$

which are all satisfied by

$$N=\alpha, \qquad M=-\beta.$$

Hence the differential equation becomes

$$\alpha d\beta-\beta d\alpha=0,$$

the integral of which is

$$\frac{\beta}{\alpha}=\text{constant};$$

and therefore, as before,

$$\frac{x_2x_4+x_1x_3}{x_2+x_3}=\text{constant}.$$

It should be noted that, in case two of the four equations which are all satisfied by θ are identical, those two may not be taken as the equations which are used to find α and β.

It will be found in practice (the reason for which can be seen from the theory of the equations) that the process adopted in the particular example for the integration of the simultaneous partial equations is of general application.

18. There is another method* of determining the integrating factor μ, which deals with that quantity as a solution of the simultaneous partial differential equations which it satisfies. The general typical equation is (§ 5)

$$X_n\frac{\partial\mu}{\partial x_m}-X_m\frac{\partial\mu}{\partial x_n}=\mu a_{m,n},$$

which must hold for all pair-combinations of the indices m, n; but, on account of the conditions satisfied by the coefficients X, a set of equations equivalent to the whole system and, so far as concerns μ, linearly independent of one another is constituted by

$$X_1\frac{\partial\mu}{\partial x_m}-X_m\frac{\partial\mu}{\partial x_1}=\mu a_{m,1}$$

* Collet, *Annales de l'Éc. Norm. Sup.*, 1re Sér., t. vii. (1870), pp. 59—88.

(where $m = 2, 3, \ldots, p$). Taking $\mu = e^z$ and denoting $\frac{\partial z}{\partial x_r}$ by p_r, we have the irreducible simultaneous system in the form

$$X_1 p_m - X_m p_1 + a_{1,m} = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (13).$$

The Jacobian conditions of coexistence are satisfied in virtue of the relations (7); and thus μ will be determined by any simultaneous solution of the system (13). If however we obtain a simultaneous solution of the system involving one or more arbitrary constants, it will in general be possible to deduce from that solution two different values of the integrating factor; and thence (§ 4) a solution of the original equation can be constructed.

Taking the special example discussed in Ex. 3, § 15 and retaining as the two linearly independent equations which determine z

$$X_1 p_3 - X_3 p_1 + a_{13} = 0,$$
$$X_2 p_3 - X_3 p_2 + a_{23} = 0,$$

we have on substitution for X_1, X_2, X_3, a_{13}, a_{23},

$$F_1 = (x_1^2 - x_2^2 - x_2 x_3)\, p_3 - x_1 (x_1 + x_2)\, p_1 - 2\,(x_1 + x_2) = 0,$$
$$F_2 = (x_2^2 - x_1^2 - x_1 x_3)\, p_3 - x_2 (x_1 + x_2)\, p_2 - 2\,(x_1 + x_2) = 0,$$

after the removal of factors x_2 and x_1 respectively.

The equations subsidiary to the integration of $F_1 = 0$ by Jacobi's process are

$$\frac{dx_1}{x_1 (x_1 + x_2)} = \frac{dx_2}{0} = \frac{dx_3}{-(x_1^2 - x_2^2 - x_2 x_3)} = \ldots\ldots$$
$$= \frac{dp_3}{-x_2 p_3}.$$

A combination of the first three of these fractions gives as their common value

$$\frac{dx_1 + dx_2 + dx_3}{x_1 x_2 + x_2^2 + x_2 x_3},$$

so that we have an integral of the system given by

$$F_3 = p_3 (x_1 + x_2 + x_3) = a,$$

where a is an arbitrary constant. It is easy to verify that

$$(F_1, F_3) = 0, \qquad (F_2, F_3) = 0;$$

and therefore $F_3 = a$ is the common solution.

Solving, we have

$$x_1 p_1 + 2 = a\, \frac{x_1^2 - x_2^2 - x_2 x_3}{(x_1 + x_2 + x_3)(x_1 + x_2)} = a \left\{ \frac{x_1}{x_1 + x_2 + x_3} + \frac{x_1}{x_1 + x_2} - 1 \right\}$$

$$x_2 p_2 + 2 = a\,\frac{x_2^2 - x_1^2 - x_1 x_3}{(x_1+x_2+x_3)(x_1+x_2)} = a\left\{\frac{x_2}{x_1+x_2+x_3} + \frac{x_2}{x_1+x_2} - 1\right\},$$

$$p_3 = \frac{a}{x_1+x_2+x_3};$$

and hence

$$\begin{aligned} dz &= p_1 dx_1 + p_2 dx_2 + p_3 dx_3 \\ &= -(a+2)\left(\frac{dx_1}{x_1} + \frac{dx_2}{x_2}\right) + a\left(\frac{dx_1+dx_2+dx_3}{x_1+x_2+x_3} + \frac{dx_1+dx_2}{x_1+x_2}\right), \end{aligned}$$

so that

$$\mu = e^z = c\,\frac{\{(x_1+x_2+x_3)(x_1+x_2)\}^a}{(x_1 x_2)^{a+2}},$$

where a is an arbitrary constant.

A special value of μ, say μ', due to $a=0$ is

$$\mu' = \frac{c}{x_1^2 x_2^2};$$

and therefore a solution of the original equation is

$$\frac{\mu}{\mu'} = \text{constant},$$

which can at once be transformed to the earlier form

$$\frac{(x_1+x_2+x_3)(x_1+x_2)}{x_1 x_2} = \text{constant}.$$

19. There is a method somewhat similar in process to Euler's (§ 12) due to Natani*; it will be sufficiently illustrated by taking the form

$$Xdx + Ydy + Zdz + Udu = 0 \quad\text{..............}\quad (A),$$

the conditions of integrability being supposed satisfied so that the integral equivalent consists of a single equation.

First, treating z and u as constant we obtain an integral of

$$Xdx + Ydy = 0$$

in a form

$$\psi(x, y, z, u) = \text{constant};$$

and the integral equivalent of the original equation is then of the form

$$f(\psi, z, u) = a\,\dagger$$

or, say

$$\psi = \phi(z, u, a) \quad\text{..........................}\quad (i),$$

* "Ueber totale und partielle Differentialgleichungen," *Crelle*, t. lviii. (1860), pp. 301—328, § 2.

† It is at this point that the assumption of integrability is tacitly made in Natani's process.

where ϕ does not explicitly involve x or y and is therefore not modified in form by any special assumption about these variables. Putting then x zero and denoting by y_1 the corresponding value of y, we have

$$\psi(x, y, z, u) = \psi(0, y_1, z, u) \quad \text{..............(ii)},$$

and then

$$\psi(0, y_1, z, u) = \phi(z, u, a) \quad \text{..................(iii)}$$

is the integral equivalent of

$$Y_1 dy_1 + Z_1 dz + U_1 du = 0 \quad \text{..................(B)},$$

where Y_1, Z_1, U_1 are the values of Y, Z, U for $y = y_1$, $x = 0$.

Secondly, treating u as constant, we obtain an integral of

$$Y_1 dy_1 + Z_1 dz = 0$$

in a form

$$\chi(y_1, z, u) = \text{constant};$$

and the integral equivalent of (B) is of the form

$$F(\chi, u) = \text{constant},$$

or say

$$\chi = \theta(u, b) \quad \text{..........................(iv)},$$

where θ does not explicitly involve y_1 or z and so is not altered in form by any special assumption as to these variables. Putting then y_1 zero and denoting by z_2 the corresponding value of z we have

$$\chi(y_1, z, u) = \chi(0, z_2, u) \quad \text{.....................(v)},$$

and then

$$\chi(0, z_2, u) = \theta(u, b) \quad \text{.....................(vi)}$$

is the integral equivalent of

$$Z_2 dz_2 + U_2 du = 0 \quad \text{.......................(C)},$$

where Z_2, U_2 are the values of Z_1, U_1 for $z = z_2$, $y_1 = 0$, i.e., are the values of Z, U for $z = z_2$, $y = 0$, $x = 0$.

Now (C), when integrated, gives z_2 as a function of u. When this value of z_2 is substituted in (v), the right-hand side becomes $\theta(u, b)$; and so the equation changes to (iv), which gives the value of y_1 as a function of z and u. When this value of y_1 is substituted in (ii), the right-hand side becomes $\phi(z, u, \text{const.})$, and so the equation changes to (i), i.e., changes to the equation which is the integral equivalent of (A) which is thus obtained.

All the steps prove possible, on account of the satisfaction of the conditions of integrability; and the functions ψ and χ are

obtained from the integration of (simplified) binomial special forms of (A) and (B).

It is evident that the difference of Natani's method from Euler's lies in the introduction of the postulates which lead to equations like (ii) and (v), and that the process will apply to equations in any number of variables. And if, in special examples, it should not prove convenient to assign a zero value to a variable in order to derive an equation such as (B) from (A), or (C) from (B), it is permissible to assign a non-zero constant value to the variable for that purpose; the only other change is that the coefficients in a general case would be less simple than they are when the normal zero-substitution is made.

Ex. 1. As a simple example, consider

$$z(y+z)\,dx+z(u-x)\,dy+y(z-u)\,dz+y(y+z)\,du=0$$

(see § 15, Ex. 1). First taking u and z as constant we have

$$\psi(x, y, z, u)=\frac{y+z}{u-x},$$

so that

$$\psi(0, y_1, z, u)=\frac{y_1+z}{u}=\frac{y+z}{u-x}.$$

The next equation is

$$zudy_1-y_1udz=0,$$

so that

$$\chi(y_1, z, u)=\frac{z}{y_1}$$

and therefore

$$\chi(1, z_2, u)=z_2=\frac{z}{y_1}.$$

The last equation is

$$-udz_2+(1+z_2)\,du=0,$$

so that

$$\frac{1+z_2}{u}=\text{constant},$$

or

$$z_2=cu-1.$$

Hence

$$y_1=\frac{z}{z_2}=\frac{z}{cu-1},$$

and therefore

$$\frac{y+z}{u-x}=\frac{y_1+z}{u}=\frac{cz}{cu-1},$$

so that

$$\frac{uy+xz}{y+z}=\frac{1}{c}=\text{constant},$$

agreeing with the former result.

Ex. 2. In order to have a direct comparison of the two methods due respectively to Euler and to Natani, we may compare them when applied to the equation

$$Xdx+Ydy+Zdz=0,$$

supposed to be integrable.

Euler takes

$$Xdx+Ydy=Md\psi$$

where ψ is a function of x, y, z determined in the first instance on the supposition that z is constant; and his second equation is then

$$d\psi+\left(\frac{Z}{M}-\frac{\partial\psi}{\partial z}\right)dz=0,$$

where $\dfrac{Z}{M}-\dfrac{\partial\psi}{\partial z}$ is a function of z and ψ alone, on account of the condition of integrability. The integral of the differential equation is then of the form

$$\psi=f(z, c),$$

where c is an arbitrary constant.

Now, with Natani,

$$\psi(x, y, z)=\psi(0, y_1, z)=\Psi(y_1, z)$$

say. Let M_1 be the value of M for $x=0$, $y=y_1$; so that, if $\theta(z, \psi)$ be the value of $\dfrac{Z}{M}-\dfrac{\partial\psi}{\partial z}$, it follows that $\theta(z, \Psi)$ is the value of $\dfrac{Z_1}{M_1}-\dfrac{\partial\Psi}{\partial z}$, since x and y do not explicitly occur. But, because $\psi=\Psi$, we have $\theta(z, \psi)=\theta(z, \Psi)$; and therefore

$$\frac{Z}{M}-\frac{\partial\psi}{\partial z}=\frac{Z_1}{M_1}-\frac{\partial\Psi}{\partial z}.$$

Also $d\psi=d\Psi$; hence Euler's second equation becomes

$$d\Psi+\left(\frac{Z_1}{M_1}-\frac{\partial\Psi}{\partial z}\right)dz=0,$$

i.e.

$$\frac{\partial\Psi}{\partial y_1}dy_1+\frac{Z_1}{M_1}dz=0.$$

Further we have

$$\frac{\partial\psi}{\partial y}=\frac{Y}{M},$$

and therefore

$$\frac{\partial\Psi}{\partial y_1}=\frac{Y_1}{M_1};$$

and hence Euler's second equation becomes

$$Y_1dy_1+Z_1dz=0,$$

i.e., it becomes Natani's second equation. Since the integral of Euler's second equation (and so of the original equation) is

$$\psi=f(z, c),$$

the integral of Natani's second equation is

$$\Psi=f(z, c),$$

and therefore the integral of the original equation is, by Natani's method,

$$\psi(x, y, z)=\Psi(y_1, z)=f(z, c).$$

20. A method has been given by du Bois-Reymond* for obtaining an integral equivalent of $\Omega = Xdx + Ydy + Zdz = 0$. But nowhere in the exposition of the method does the condition of integrability essentially appear; and, whether $\Omega = 0$ be integrable or not, the method leads to a single equation, which is declared to be of the proper form when the equation is integrable.

The method, geometrically stated, is as follows:—Whatever be the locus represented by $\Omega = 0$, we pass from a point P_0, whose coordinates are x_0, y_0, z_0, along that part of the locus which lies on an arbitrary surface

$$a = \chi(x, y, z)$$

to some other point P_1, whose coordinates are x_1, y_1, z_1. From this point P_1, we move along that part of the locus which lies on another arbitrary surface

$$b = \chi_1(x, y, z)$$

to some other point P, whose coordinates are x, y, z.

Now

$$\Omega = 0, \quad a = \chi, \quad 0 = d\chi$$

can be used to construct an equation of the first order in two variables only; of this there will be an integral, say

$$f(x, y, z, a) = c_1,$$

so that we have

$$\chi(x_0, y_0, z_0) = a = \chi(x_1, y_1, z_1),$$
$$f(x_0, y_0, z_0, a) = c_1 = f(x_1, y_1, z_1, a).$$

Similarly

$$\Omega = 0, \quad b = \chi_1, \quad 0 = d\chi_1$$

can be used to construct an equation of the first order in two variables only, with an integral of the form

$$g(x, y, z, b) = c_2,$$

so that we have

$$\chi_1(x_1, y_1, z_1) = b = \chi_1(x, y, z),$$
$$g(x_1, y_1, z_1, b) = c_2 = g(x, y, z, b).$$

From these eight equations the seven quantities $x_1, y_1, z_1, a, b, c_1, c_2$ can be eliminated; and the result will be

$$\Phi(x, y, z, x_0, y_0, z_0) = 0,$$

* "Ueber die Integration linearer totaler Differentialgleichungen denen durch ein Integral Genüge geschieht," *Crelle*, t. lxx. (1869), pp. 299—318.

which, if the equation $\Omega = 0$ be integrable, is assumed to take the form

$$\phi(x, y, z) = \phi(x_0, y_0, z_0) = \text{constant}.$$

This is the equation of the locus of P, and thus is the integral equivalent of $\Omega = 0$ the differential equation of that locus.

The assumption just made as to the form of the final equation is not proved, though it can be verified in special instances. Moreover no proof is given that, if the assumption as to the form be generally true, the result is independent of any peculiarity in the form of the subsidiary equations $\chi = a$, $\chi_1 = b$ which may be adopted.

Modifications of the method, subject to the same criticisms, are given (l. c.), and a generalisation to equations in n variables*.

Ex. 1. When the method is applied to the integrable equation

$$2(y+z)\,dx + x\,dy + x\,dz = 0$$

with the subsidiary equations

$$\chi = \frac{y}{x}, \qquad \chi_1 = \frac{z}{x},$$

it leads (correctly) to the result

$$x^2(y+z) = x_0^2(y_0+z_0),$$

a single equation of the assumed type.

But when applied to the non-integrable equation

$$x\,dy + y\,dx + x\,dz = 0$$

with the subsidiary equations

$$\chi = \frac{y}{x}, \qquad \chi_1 = \frac{z}{x},$$

it leads to

$$(2y+z)^2 = (2y_0+z_0)\left(2y\frac{x_0}{x} + z_0\right),$$

and when applied to the same equation with the subsidiary equations

$$\chi = z, \qquad \chi_1 = y,$$

it leads to

$$\frac{z-z_0}{y} = \log\frac{x_0 y_0}{xy};$$

in each case (incorrectly) to a single equation not, however, of the assumed type.

Ex. 2. It is easy to see that the practical rule given in du Bois-Reymond's method comes to be the same as the process in Natani's method (§ 19), when the subsidiary equations are

$$\chi = z = a, \qquad \chi_1 = y = 0.$$

* For other criticisms, see Weiler, *Schlöm. Zeitschr.*, t. xx. (1875), pp. 80—83; du Bois-Reymond, *Math. Ann.*, t. xii. (1877), pp. 123—131.

21. The derivation of the conditions of exact integrability of an ordinary differential equation of the nth order (or of a differential expression involving derivatives of a single dependent variable with regard to a single independent variable) is sometimes made to depend upon the theory of integration of an expression, exact in the sense of the foregoing chapter. As however the connection is not immediate and this method is not the principal method, it will be sufficient here to give the following references to some of the writers on the subject, in whose memoirs references to Euler, Lagrange, Lexell, and Condorcet, will be found:

SARRUS; *Comptes Rendus*, t. i. (1835), pp. 115—117: t. xxviii. (1849), pp. 439—442; *Liouville*, t. xiv. (1849), pp. 131—134.

DIRKSEN; *Abh. d. Kön. Akad. d. Wiss. zu Berlin* (1836), pp. 79—98.

BERTRAND; *Journ. de l'Éc. Poly.*, t. xvii. (1841), pp. 249—275; *Liouville*, t. xiv. (1849), pp. 123—130.

RAABE; *Crelle*, t. xxxi. (1846), pp. 181—212.

JOACHIMSTAHL; *Crelle*, t. xxxiii. (1846), pp. 95—116.

STOFFEL ET BACH; *Liouville*, 2^{me} Sér., t. vii. (1862), pp. 49—61.

IMSCHENETSKY; *Grun. Arch.*, t. l. (1869), pp. 278—474; especially § 26.

PUJET; *Comptes Rendus*, t. lxxxii. (1876), pp. 740—743.

WINCKLER; *Wiener Sitzungsb.*, t. lxxxviii., Abth. ii. (1883), pp. 820—834.

MISCELLANEOUS EXAMPLES.

1. Let P, Q, R be any functions of three independent variables x, y, z; and with a point A (x, y, z) let the plane $(X-x)P+(Y-y)Q+(Z-z)R=0$ be associated. Shew that there are two directions in this plane such that, if a point $B(x+dx,\ y+dy,\ z+dz)$ be taken consecutive to A lying in the plane in either direction, the intersection of the planes associated with A and with B is the line AB. Shew also that, if the relation

$$\begin{vmatrix} \frac{\partial P}{\partial x}, & \frac{\partial P}{\partial y}, & \frac{\partial P}{\partial z}, & P \\ \frac{\partial Q}{\partial x}, & \frac{\partial Q}{\partial y}, & \frac{\partial Q}{\partial z}, & Q \\ \frac{\partial R}{\partial x}, & \frac{\partial R}{\partial y}, & \frac{\partial R}{\partial z}, & R \\ P, & Q, & R, & 0 \end{vmatrix} = 0$$

be identically satisfied, then the system of planes associated with all possible points A envelopes a surface.

(VOSS.)

2. In connection with the last example shew that, if for the equation $Pdx+Qdy+Rdz=0$ the condition of integrability

$$G=P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0$$

be identically satisfied, then the envelope of the associated planes is in its general form an integral of the differential equation; and that, at any point of contact of a plane with the surface, the two directions indicated are those of the inflexional tangents to the surface.

In the case when $G=0$ is not identically satisfied, discuss the relation of the surface $G=0$ with the differential equation.

(Voss.)

3. Prove that the equation

$$(x_2^2-x_3x_1)\,dx_1+(x_3^2-x_1x_2)\,dx_2+(x_1^2-x_2x_3)\,dx_3=0$$

is exact; and obtain its integral by Bertrand's method (or otherwise) in the form

$$(x_1+x_2+x_3)^{\omega^2}(x_1+\omega^2x_2+\omega x_3)^{\omega}(x_1+\omega x_2+\omega^2x_3)=\text{constant}.$$

4. Shew that the total differential equations

(i) $x(y-1)(z-1)\,dx+y(z-1)(x-1)\,dy+z(x-1)(y-1)\,dz=0,$

(ii) $dx+\dfrac{x}{y}\,dy-\dfrac{x}{2z}\,dz=0,$

(iii) $dx+\dfrac{x}{2y}\,dy-\dfrac{zu}{xy}\,dz-\dfrac{z^2}{2xy}\,du=0,$

(iv) $dx+\dfrac{x}{y\log yz}\,dy+\dfrac{x}{z\log yz}\,dz+\dfrac{x}{v}\cot\dfrac{u}{v}\left(du-\dfrac{u}{v}\,dv\right)=0,$

all satisfy the conditions of integrability; and obtain the respective integral equations which are equivalent to these differential relations.

(Collet.)

5. Integrate the following equations which can, on multiplication by a factor, be made exact:—

(i) $(yz+z^2)\,dx-xzdy+xydz=0,$

(ii) $(y+z)^2\,dx+z^2dy+y^2dz=0,$

(iii) $z(1-z^2)\,dx+zdy-(x+y+xz^2)\,dz=0,$

(iv) $(1+yz)^2\,dx-z^2dy+dz=0,$

(v) $H(y,z,u)\,dx+H(z,u,x)\,dy+H(u,x,y)\,dz+H(x,y,z)\,du=0,$

where in the last $H(a,b,c)$ denotes the sum of the homogeneous products of a, b, c of two dimensions.

6. Obtain the primitive of

$$\begin{vmatrix} dx, & dy, & dz, & 0 \\ x, & y, & z, & 1 \\ A_0, & A_1, & A_2, & A_3 \\ B_0, & B_1, & B_2, & B_3 \end{vmatrix}=0$$

(the generalisation of Hesse's equation), where the quantities A and B are linear non-homogeneous functions of x, y, z, in the form

$$u_0^{c_0} u_1^{c_1} u_2^{c_2} u_3^{c_3} = \text{constant},$$

where u_0, u_1, u_2, u_3 are properly determined linear functions of x, y, z and c_0, c_1, c_2, c_3 are constants.

(Pittarelli.)

7. Shew that, if the equation

$$P_1 dx_1 + P_2 dx_2 + \ldots\ldots + P_n dx_n = 0$$

be deducible from a single integral equation and be such that $P_1, P_2, \ldots, P_n$ are homogeneous functions all of the same order, then

$$(P_1 x_1 + P_2 x_2 + \ldots\ldots + P_n x_n)^{-1}$$

is an integrating factor.

Discuss the case in which this integrating factor takes the form $1 \div 0$; and apply the result to integrate

$$(y+z)^2\, dx - x(y+2z)\, dy - xz dz = 0.$$

(Fais.)

8. Shew that, if the coefficients in $\sum_{m=0}^{n-1} X_m dx_m$ be rational, then the substitutions (for $k=0, 1, \ldots, n-1$)

$$x_k = (z_0 + \omega^k z_1 + \omega^{2k} z_2 + \ldots\ldots + \omega^{nk-k} z_{n-1})^n,$$

where ω is a primitive nth root of unity, transform the differential expression into

$$\sum_{m=0}^{n-1} \phi(z_{m+1}, z_{m+2}, \ldots\ldots, z_{m-1})\, dz_m;$$

and prove that the necessary and sufficient conditions of integrability of the expression reduce themselves to the $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$ equations

$$\frac{\partial \phi(z_0, z_1, \ldots, z_{n-1})}{\partial z_r} = \frac{\partial \phi(z_r, z_{r+1}, \ldots, z_{r-1})}{\partial z_0}$$

for $r = 1, 2, 3, \ldots \frac{1}{2}n$ or $\frac{1}{2}(n-1)$.

(Kronecker.)

9. Prove that conditions, necessary and sufficient to ensure that

$$p_1 dx_1 + p_2 dx_2 + \ldots\ldots + p_n dx_n$$

is an exact differential, are

(i) that the equations

$$\frac{\partial p_1}{\partial x_j} = \frac{\partial p_j}{\partial x_1} \; (j = 2, \ldots, n)$$

are satisfied, and

(ii) that $p_2 dx_2 + \ldots\ldots + p_n dx_n$ is an exact differential when x_1 is made constant.

(Laurent.)

3—2

10. If the equation

$$X_1dx_1 + X_2dx_2 + \ldots\ldots + X_pdx_p = 0$$

satisfy all the conditions that it should be capable of representation by a single integral equation and if it be reduced in succession to the forms

$$\begin{aligned} du_1 + Y_3dx_3 + Y_4dx_4 + \ldots\ldots + Y_pdx_p &= 0, \\ du_2 + Z_4dx_4 + \ldots\ldots + Z_pdx_p &= 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ du_{p-2} + T_{p-2}dx_p &= 0, \\ du_{p-1} &= 0, \end{aligned}$$

then an integrating factor of the original equation is

$$\frac{1}{X_1}\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial u_1}\ldots\ldots\frac{\partial u_{p-2}}{\partial u_{p-3}}\frac{\partial u_{p-1}}{\partial u_{p-2}}.$$

CHAPTER II.

SYSTEMS OF EXACT EQUATIONS.

22. WHEN we have a system of n equations in $m+n$ variables

$$\phi_r(u_1, u_2, \ldots, u_n, x_1, \ldots, x_m) = a_r,$$

where a_r is a constant and r may have the values $1, 2 \ldots, n$, they can be considered as determining the n variables $u_1, u_2, \ldots, u_n$ in terms of the m variables $x_1, \ldots, x_m$, provided there is no functional relation among the n functions ϕ considered as involving the n variables u.

All variations of the variables are connected by n equations of the form

$$\sum_s \frac{\partial\phi}{\partial u_s} du_s + \sum_t \frac{\partial\phi}{\partial x_t} dx_t = 0.$$

When they are solved, they give the variations du of the dependent variables in terms of the variations dx of the independent variables by means of other n equations

$$du_s = \sum_{t=1}^{m} U_{s,t}\, dx_t \ldots (s = 1, 2, \ldots, n) \ldots\ldots\ldots\ldots \text{(I)},$$

where the coefficients $U_{s,t}$ are (or may be) functions of all the variables u and x; and the equations (I) are definitely and uniquely derivable from the earlier set because

$$\frac{\partial(\phi_1, \ldots\ldots, \phi_n)}{\partial(u_1, \ldots\ldots, u_n)},$$

which is the determinant of the coefficients of the du's in that earlier set, does not vanish on account of the hypothesis made regarding the functions ϕ.

Moreover, since the equations (I) are derived from the earlier set, the substitution there of the values of du as given by (I) will lead to identities; hence, for each of the n functions ϕ, the equation

$$\sum_{t=1}^{m}\left(\sum_{s=1}^{n}\frac{\partial\phi}{\partial u_s}U_{s,t}\right)dx_t+\sum_{t=1}^{m}\frac{\partial\phi}{\partial x_t}dx_t=0$$

is identically satisfied. It follows that the coefficient of each of the differential elements dx must vanish; and therefore ϕ satisfies the m differential equations

$$\Delta_t\phi=\frac{\partial\phi}{\partial x_t}+\sum_{s=1}^{n}U_{s,t}\frac{\partial\phi}{\partial u_s}=0\;\ldots\ldots\ldots\ldots\ldots\ldots\text{(II)}$$

for $t=1, 2, \ldots, m$.

23. It follows, just as in the case of a single equation, that any functional combination of the quantities ϕ is a solution of the equations (I) or (II). For, so far as regards the equations (I), they are equivalent to

$$d\phi_1=0,\ d\phi_2=0,\ \ldots\ldots,\ d\phi_n=0,$$

and therefore, if ψ be a function of $\phi_1, \phi_2, \ldots, \phi_n$, we have

$$\begin{aligned}d\psi&=\sum_{r=1}^{n}\frac{\partial\psi}{\partial\phi_r}d\phi_r\\&=0\end{aligned}$$

in virtue of equations (I); or ψ is a solution. And, so far as regards the equations (II), we have

$$\begin{aligned}\Delta_t\psi&=\sum_{r=1}^{n}\frac{\partial\psi}{\partial\phi_r}\Delta_t\phi_r\\&=0\end{aligned}$$

in virtue of equations (II); or ψ is a solution.

Conversely, every solution of the system (I) or the system (II) can be expressed as a function of the n independent solutions $\phi_1, \phi_2, \ldots, \phi_n$. For, taking it to be ψ, it is some function of $u_1, u_2, \ldots, u_n$ and of the independent variables $x_1, x_2, \ldots, x_m$; and determining $u_1, u_2, \ldots, u_n$ in terms of the solutions ϕ and of the variables x we then have ψ in the form

$$\psi=\psi(\phi_1, \phi_2, \ldots, \phi_n, x_1, x_2, \ldots, x_m).$$

So far as regards (I), we have $d\psi=0$, for ψ is a solution; and equations (I) give $d\phi_1=0, \ldots, d\phi_n=0$, for the quantities ϕ are solutions; hence

$$0=\frac{\partial\psi}{\partial x_1}dx_1+\frac{\partial\psi}{\partial x_2}dx_2+\ldots\ldots+\frac{\partial\psi}{\partial x_m}dx_m.$$

Now the quantities $x_1, x_2, \ldots, x_m$ are by hypothesis independent and there can therefore be no relation among their variations. Hence the foregoing equations can only be satisfied by

$$\frac{\partial\psi}{\partial x_1}=0,\ \frac{\partial\psi}{\partial x_2}=0,\ \ldots\ldots,\ \frac{\partial\psi}{\partial x_m}=0\,;$$

or ψ in its later form is explicitly independent of $x_1, x_2, \ldots, x_m$ and can thus be expressed as a function of $\phi_1, \ldots, \phi_n$. So far as regards (II), we have, taking ψ in the same form, the equations

$$0=\Delta_t\psi=\sum_{r=1}^{n}\left(\frac{\partial\psi}{\partial\phi_r}\,\Delta_t\phi_r\right)+\frac{\partial\psi}{\partial x_t},$$

for the quantities u occur only in the quantities ϕ: hence, as $\Delta_t\phi_r=0$, we have

$$\frac{\partial\psi}{\partial x_t}=0$$

for each of the values $t=1, 2, \ldots, m$; and thus we are led to the same conclusion as before.

24. Multiplying the equations (I) by $\lambda_{1,s}, \lambda_{2,s}, \ldots, \lambda_{r,s}$ where $\lambda_{r,s}=\dfrac{\partial\phi_r}{\partial u_s}$ and adding, we have

$$d\phi_r=0,$$

an equation in an integrable form. Such a system of quantities may be called a set of integrating factors; there must evidently be n independent sets of integrating factors, each set subsidiary to the derivation of an integrable equation; and the determinant of the n sets is not zero, because the Jacobian of the functions ϕ with regard to the variables u is not zero.

Now let ψ be any other integral of the system of equations so that, as just proved,

$$\psi=\psi(\phi_1, \phi_2, \ldots\ldots, \phi_n)\,;$$

and let $\rho_1, \rho_2, \ldots, \rho_n$ be the set of integrating factors subsidiary to the derivation of $d\psi=0$. Then we have

$$\rho_r=\frac{\partial\psi}{\partial u_r}=\frac{\partial\psi}{\partial\phi_1}\lambda_{1,r}+\frac{\partial\psi}{\partial\phi_2}\lambda_{2,r}+\ldots\ldots+\frac{\partial\psi}{\partial\phi_n}\lambda_{n,r}$$

for $r=1, 2, \ldots, n$. The determinant of the right-hand sides, considered as linear in the quantities $\dfrac{\partial\psi}{\partial\phi}$, is a non-vanishing

quantity being made up of the n independent sets of integrating factors; so that the n equations can be solved to give the quantities $\dfrac{\partial\psi}{\partial\phi}$ in the forms

$$\frac{\partial\psi}{\partial\phi_1}=\frac{M_1}{M},\ \frac{\partial\psi}{\partial\phi_2}=\frac{M_2}{M},\ \ldots\ldots,\ \frac{\partial\psi}{\partial\phi_n}=\frac{M_n}{M}.$$

The value of M is

$$\frac{\partial(\phi_1, \phi_2, \ldots\ldots, \phi_n)}{\partial(u_1, u_2, \ldots\ldots, u_n)}$$

and that of M_r is

$$(-1)^r\frac{\partial(\phi_1, \ldots\ldots, \phi_{r-1}, \psi, \phi_{r+1}, \ldots\ldots, \phi_n)}{\partial(u_1, u_2, \ldots\ldots, u_n)},$$

so that all the quantities M are of the same kind, viz., Jacobians of systems of n solutions of the original equations. Such a quantity as M, or M_r, is called, after Jacobi, a *multiplier*: it is, from its origin, a non-vanishing magnitude.

Now unless the Hessian of ψ, considered as a function of $\phi_1, \ldots, \phi_n$, vanishes, the quantities $\dfrac{\partial\psi}{\partial\phi}$ are independent of one another, so that there is no identical relation between them involving only constant coefficients. The effect of this independence of the quantities M is to exclude the existence among the $n+1$ quantities M of any homogeneous relation of any degree with constant coefficients.

If then we suppose that no one of the quantities $\dfrac{\partial\psi}{\partial\phi}$ is a constant or zero, each of them will be a functional combination of the quantities ϕ and so will be a solution of the system of equations; and, as they are supposed to be independent functional combinations of the quantities ϕ, it therefore follows that the n quantities $\dfrac{\partial\psi}{\partial\phi}$ will constitute a complete system of integrals, similar to the system of integrals $\phi_1, \phi_2, \ldots, \phi_n$.

We may now draw some inferences:—

(A) *When a multiplier M of the equations* (I) *is known, then every other multiplier is of the form $M\Phi$, where $\Phi=$ constant is a solution of the equations.*

For every other multiplier is a Jacobian of a system of solutions; the quotient of this Jacobian by M is the Jacobian of the system of solutions with regard to $\phi_1, \ldots, \phi_n$ and therefore, as it is a function of $\phi_1, \phi_2, \ldots, \phi_n$, it is a solution of the equations. Hence, conversely:

(A′) *If two independent multipliers have been obtained which have not a constant ratio, their quotient is a solution of the equations* (I) *or* (II).

Also

(B) *If $n+1$ multipliers have been obtained such that there exists among them no homogeneous relation of any order, then the n quotients of any n of them by the remaining one form a complete system of integrals of the differential equations.*

And, as can easily be proved by similar considerations:—

(C) *If $p+n+1$ multipliers have been obtained, then there exist among them at least p identical relations, the coefficients in which are constants.*

25. Now we can deduce from a well-known theorem in determinants, due to Jacobi*, the partial differential equations which these multipliers satisfy. If there be $n+1$ functions (say $\phi_1, \ldots, \phi_n$ the foregoing functions and ϕ any other function) which involve $n+1$ variables (say $u_1, \ldots, u_n$ and x any other variable) and if

$$\frac{\partial(\phi, \phi_1, \ldots\ldots, \phi_n)}{\partial(x, u_1, \ldots\ldots, u_n)} = A\frac{\partial\phi}{\partial x} + A_1\frac{\partial\phi}{\partial u_1} + \ldots\ldots + A_n\frac{\partial\phi}{\partial u_n},$$

then the quantities A are such that

$$\frac{\partial A}{\partial x} + \frac{\partial A_1}{\partial u_1} + \ldots\ldots + \frac{\partial A_n}{\partial u_n} = 0.$$

This theorem is applicable to the set of functions under consideration, when we make x the same as any one of the variables $x_1, x_2, \ldots, x_m$. We have A given by the quantity M; and we have, for $x = x_t$,

$$A_r = (-1)^r \frac{\partial(\phi_1, \phi_2, \ldots\ldots\ldots\ldots, \phi_n)}{\partial(x, u_1, \ldots, u_{r-1}, u_{r+1}, \ldots, u_n)},$$

* *Crelle*, t. xxvii. (1844), p. 21; *Ges. Werke*, t. iv., p. 328.

where, in the denominator, u_r does not occur. But, by (II),

$$\frac{\partial\phi}{\partial x_t} = -\sum_{s=1}^{n} U_{s,t}\frac{\partial\phi}{\partial u_s},$$

so that
$$A_r = (-1)^{r+1}\frac{\partial(\phi_1, \phi_2, \ldots\ldots, \phi_n)}{\partial(u_r, u_1, \ldots\ldots, u_n)} U_{r,t}$$

where u_r does not occur in the sequence $u_1, \ldots, u_n$; and thus

$$A_r = (-1)^{r+1}.(-1)^{r-1}\frac{\partial(\phi_1, \phi_2, \ldots\ldots, \phi_n)}{\partial(u_1, u_2, \ldots\ldots, u_n)} U_{r,t}$$

$$= \frac{\partial(\phi_1, \phi_2, \ldots\ldots, \phi_n)}{\partial(u_1, u_2, \ldots\ldots, u_n)} U_{r,t},$$

where $u_1, u_2, \ldots, u_n$ are now the complete set of all the variables u. Hence

$$A_r = MU_{r,t};$$

and therefore substituting for $A_r (r = 1, 2, \ldots, n)$ and for A we have the equation

$$\frac{\partial M}{\partial x_t} + \frac{\partial(MU_{1,t})}{\partial u_1} + \frac{\partial(MU_{2,t})}{\partial u_2} + \ldots\ldots + \frac{\partial(MU_{n,t})}{\partial u_n} = 0.$$

The dependent variable M in this equation is the non-evanescent Jacobian of n solutions of the equations (I) or (II) with regard to $u_1, \ldots, u_n$; and therefore the same equation is satisfied when the function ψ is substituted for $\phi_1, \phi_2, \ldots, \phi_n$ in turn in this Jacobian, that is, the other multipliers $M_1, \ldots, M_n$ also satisfy the equation. Hence, by propositions (B) and (C), every multiplier satisfies this equation; and, therefore, as the equation holds for the values $t = 1, 2, \ldots, m$, *the multipliers of the system* (I) *or* (II) *are determined by the equations*

$$\left.\begin{aligned}
\nabla_1\theta &= \frac{\partial\theta}{\partial x_1} + \frac{\partial(\theta U_{1,1})}{\partial u_1} + \frac{\partial(\theta U_{2,1})}{\partial u_2} + \ldots\ldots + \frac{\partial(\theta U_{n,1})}{\partial u_n} = 0 \\
\nabla_2\theta &= \frac{\partial\theta}{\partial x_2} + \frac{\partial(\theta U_{1,2})}{\partial u_1} + \frac{\partial(\theta U_{2,2})}{\partial u_2} + \ldots\ldots + \frac{\partial(\theta U_{n,2})}{\partial u_n} = 0 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\nabla_m\theta &= \frac{\partial\theta}{\partial x_m} + \frac{\partial(\theta U_{1,m})}{\partial u_1} + \frac{\partial(\theta U_{2,m})}{\partial u_2} + \ldots\ldots + \frac{\partial(\theta U_{n,m})}{\partial u_n} = 0
\end{aligned}\right\} \ldots\text{(III)}.$$

Ex. A verification of (A′) in this paragraph can be deduced at once from (III). For if θ and $\theta\phi$ be two solutions of (III), we have

$$\begin{aligned} \nabla_t\theta &= 0, \\ 0 &= \nabla_t(\theta\phi) \\ &= \phi\nabla_t\theta + \theta\Delta_t\phi; \end{aligned}$$

and therefore ϕ, the quotient of two solutions of (III), satisfies

$$\Delta_t\phi = 0$$

for the values $t = 1, 2, \ldots, m$. Thus ϕ is a solution of (II), the original system of equations.

26. Suppose now that, instead of deducing differential equations from a set of given integral equations of the type considered, it is required to obtain a system of integral equations equivalent to a given system of independent differential equations.

The most general form of the latter is

$$\sum_{s=1}^{n} X_{r,s}du_s + \sum_{t=1}^{m} Y_{r,t}dx_t = 0 \quad (r = 1, 2, \ldots, n),$$

the coefficients X and Y being functions of the variables u and x. Since any linear combination of these equations will be satisfied by the same integral equations as the foregoing set, they may be replaced by the set

$$du_r = \sum_{t=1}^{m} U_{r,t}dx_t \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathrm{I})$$

obtained by solving the system for $du_1, \ldots, du_n$; and there are n equations in this set, all the coefficients U being functions of x and u. It is assumed that the equations can be thus expressed—an assumption which implies that the determinant of the coefficients X does not vanish and which is justified by the supposition that the system of equations is independent; and the form (I) will be considered the canonical form of such systems of equations.

In order that a system of equations such as (I) may be replaced by a system of integral equations, which suffice to determine $u_1, u_2, \ldots, u_n$ as functions of independent variables $x_1, x_2, \ldots, x_m$, certain conditions must be satisfied. Instead of regarding the coefficients U as composed of derivatives of the (unknown) functions ϕ, we may regard the u's as (potentially) expressible by means of the unknown integral system in the form of explicit functions of the variables $x_1, x_2, \ldots, x_m$; and when they are so expressed they satisfy the necessary and sufficient conditions

$$\frac{\partial}{\partial x_s}\left(\frac{\partial u}{\partial x_t}\right) = \frac{\partial}{\partial x_t}\left(\frac{\partial u}{\partial x_s}\right),$$

which apply to each of the dependent variables for all combinations of s and t.

From the system (I) we have

$$\frac{\partial u_r}{\partial x_t} = U_{r,t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

for each of the indices $t\,(=1, 2, \ldots, m)$ and each of the indices $r\,(=1, 2, \ldots, n)$; and when we substitute in $U_{r,t}$ the values of those variables u which may occur, in terms of $x_1, x_2, \ldots, x_m$, the resulting value of $\dfrac{\partial u_r}{\partial x_t}$ is the same as in the above condition. Denoting this value by $\left[\dfrac{\partial u_r}{\partial x_t}\right]$, the foregoing condition is

$$\frac{\partial}{\partial x_s}\left[\frac{\partial u_r}{\partial x_t}\right] = \frac{\partial}{\partial x_t}\left[\frac{\partial u_r}{\partial x_s}\right].$$

The aggregate of these conditions, necessary and sufficient for the integral equations, is certainly necessary for the differential equations under the supposed hypothesis; it must not, without proof, be assumed sufficient for the differential equations. Now we have

$$\begin{aligned}\frac{\partial}{\partial x_s}\left[\frac{\partial u_r}{\partial x_t}\right] &= \frac{\partial U_{r,t}}{\partial x_s} + \sum_{p=1}^{n} \frac{\partial U_{r,t}}{\partial u_p}\frac{\partial u_p}{\partial x_s} \\ &= \frac{\partial U_{r,t}}{\partial x_s} + \sum_{p=1}^{n} U_{p,s}\frac{\partial U_{r,t}}{\partial u_p},\end{aligned}$$

where the quantities on the right-hand side are as they occur in (I); and similarly for the other member of the equation of condition, which therefore becomes

$$\frac{\partial U_{r,t}}{\partial x_s} + \sum_{p=1}^{n} U_{p,s}\frac{\partial U_{r,t}}{\partial u_p} = \frac{\partial U_{r,s}}{\partial x_t} + \sum_{p=1}^{n} U_{p,t}\frac{\partial U_{r,s}}{\partial u_p},$$

or, what is the same thing,

$$\frac{\partial U_{r,t}}{\partial x_s} - \frac{\partial U_{r,s}}{\partial x_t} + \sum_{p=1}^{n}\left(U_{p,s}\frac{\partial U_{r,t}}{\partial u_p} - U_{p,t}\frac{\partial U_{r,s}}{\partial u_p}\right) = 0 \quad \ldots\ldots(2).$$

This equation must be satisfied for

(i) each of the values $1, 2, \ldots, n$ of r:

(ii) each of the $\frac{1}{2}m(m-1)$ combinations of s and t from the series $1, 2, \ldots, m$;

therefore *the total number of necessary conditions represented by* (2) *is*

$$\tfrac{1}{2}nm(m-1),$$

and these conditions are independent of one another.

27. It now remains to prove that these necessary conditions are sufficient to ensure that the integral equivalent of the n differential equations (I) is constituted by a system of n integral equations independent of one another. For this purpose an inductive process will be used. The given system (I), involving n dependent and m independent variables, will be transformed by a change of variables to a similar system, involving n dependent and $m-1$ independent variables; and, of the $\tfrac{1}{2}nm(m-1)$ necessary conditions of the original system, $\tfrac{1}{2}n(m-1)(m-2)$ conditions bearing the necessary corresponding form will be shewn to survive for the transformed system.

To obtain the transformations we subsidiarily consider all the variables $x_2, \ldots, x_m$ as unchanging; the equations (I) take the forms

$$\frac{du_1}{U_{11}} = \frac{du_2}{U_{21}} = \ldots\ldots = \frac{du_n}{U_{n1}} = dx_1.$$

Let n independent integrals of these n equations be

$$\xi_1 = \text{constant},\ \ \xi_2 = \text{constant},\ \ \ldots\ldots,\ \ \xi_n = \text{constant},$$

where $\xi_1, \xi_2, \ldots, \xi_n$ are functions of $u_1, u_2, \ldots, u_n, x_1$ and of unchanging quantities that may occur in $U_{11}, U_{21}, \ldots, U_{n1}$, that is, of $x_2, \ldots, x_n$. All these functions ξ satisfy the equation

$$\frac{\partial \xi}{\partial x_1} + \sum_{r=1}^{n} U_{r1} \frac{\partial \xi}{\partial u_r} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Having thus obtained these n independent functions ξ, we use them to form equations of transformation in the shape

$$\xi_1 = v_1,\ \ \xi_2 = v_2,\ \ \ldots\ldots,\ \ \xi_n = v_n,$$

where $v_1, v_2, \ldots, v_n$ (constant so far as regards x_1) are functions of $x_2, \ldots, x_m$, now to be determined. Taking any one of the equations, say

$$\xi = v,$$

we have

$$dv = \sum_{r=1}^{n} \frac{\partial \xi}{\partial u_r} du_r + \sum_{s=1}^{m} \frac{\partial \xi}{\partial x_s} dx_s$$

$$= \sum_{s=1}^{m} \left(\frac{\partial \xi}{\partial x_s} + U_{1s} \frac{\partial \xi}{\partial u_1} + U_{2s} \frac{\partial \xi}{\partial u_2} + \ldots\ldots + U_{ns} \frac{\partial \xi}{\partial u_n} \right) dx_s,$$

after substitution for the quantities du from the equations (I). The coefficient of the element dx_1 on the right-hand side vanishes on account of the partial differential equation satisfied by ξ, so that the assumption of dv being independent of dx_1 is hereby justified. When we take

$$\frac{\partial \xi_r}{\partial x_s} + U_{1s} \frac{\partial \xi_r}{\partial u_1} + U_{2s} \frac{\partial \xi_r}{\partial u_2} + \ldots\ldots + U_{ns} \frac{\partial \xi_r}{\partial u_n} = V'_{rs} \ldots\ldots (4),$$

then we have

$$dv_r = \sum_{s=2}^{m} V'_{rs} dx_s \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathrm{I})',$$

for $r = 1, 2, \ldots, n$.

28. This is the transformed system indicated: we proceed to consider the coefficients V'_{rs}. From the n independent equations

$$\xi = v,$$

we can find the values of $u_1, u_2, \ldots, u_n$ as functions of $v_1, v_2, \ldots, v_n, x_1, x_2, \ldots, x_m$; and when these values are substituted for the u's in V'_{rs}, wherever they occur, it is changed into a function of $v_1, v_2, \ldots, v_n, x_1, \ldots, x_m$. Let V_{rs} denote this transformed value of V'_{rs}: then *V_{rs} is explicitly independent of x_1*, a result which leaves the system (I)′ involving only $m - 1$ independent variables.

The proposition just enunciated may be proved as follows. Let Θ' denote any function of $u_1, u_2, \ldots, u_n, x_1, \ldots, x_m$ and Θ the same function when substitution is made for $u_1, u_2, \ldots, u_n$ in terms of $v_1, v_2, \ldots, v_n, x_1, \ldots, x_m$; thus Θ involves $x_1, x_2, \ldots, x_m$ and $v_1 (= \xi_1), v_2 (= \xi_2), \ldots, v_n (= \xi_n)$. Then

$$\frac{\partial \Theta'}{\partial x_1} = \frac{\partial \Theta}{\partial x_1} + \sum_{r=1}^{n} \frac{\partial \Theta}{\partial \xi_r} \frac{\partial \xi_r}{\partial x_1};$$

and

$$\frac{\partial \Theta'}{\partial u_p} = \sum_{r=1}^{n} \frac{\partial \Theta}{\partial \xi_r} \frac{\partial \xi_r}{\partial u_p}$$

for $p = 1, 2, \ldots, n$; hence

$$\frac{\partial \Theta'}{\partial x_1}+\sum_{p=1}^{n} U_{p,1}\frac{\partial \Theta'}{\partial u_p}=\frac{\partial \Theta}{\partial x_1}+\sum_{r=1}^{n}\frac{\partial \Theta}{\partial \xi_r}\left\{\frac{\partial \xi_r}{\partial x_1}+\sum_{p=1}^{n} U_{p,1}\frac{\partial \xi_r}{\partial u_p}\right\}.$$

By the differential equation (3) which all the functions ξ satisfy, the coefficient of every term $\frac{\partial \Theta}{\partial \xi_r}$ on the right-hand side is zero; hence

$$\frac{\partial \Theta}{\partial x_1}=\frac{\partial \Theta'}{\partial x_1}+\sum_{p=1}^{n} U_{p,1}\frac{\partial \Theta'}{\partial u_p},$$

giving the value of $\frac{\partial \Theta}{\partial x_1}$, so far as x_1 occurs explicitly in the transformed function Θ, in terms of derivatives of Θ' the untransformed function.

Applying this general result to the coefficients in (I)′, we have

$$\frac{\partial V_{rs}}{\partial x_1}=\frac{\partial V'_{rs}}{\partial x_1}+\sum_{p=1}^{n} U_{p,1}\frac{\partial V'_{rs}}{\partial u_p}.$$

Substituting from (4) the value

$$\frac{\partial \xi_r}{\partial x_s}+\sum_{q=1}^{n} U_{q,s}\frac{\partial \xi_r}{\partial u_q}$$

for V'_{rs}, we have

$$\begin{aligned}\frac{\partial V_{rs}}{\partial x_1}=\frac{\partial^2 \xi_r}{\partial x_1\partial x_s}&+\sum_{p=1}^{n} U_{p,1}\frac{\partial^2 \xi_r}{\partial x_s\partial u_p}\\&+\sum_{q=1}^{n} U_{q,s}\left\{\frac{\partial^2 \xi_r}{\partial u_q\partial x_1}+\sum_{p=1}^{n} U_{p,1}\frac{\partial^2 \xi_r}{\partial u_q\partial u_p}\right\}\\&+\sum_{q=1}^{n}\frac{\partial \xi_r}{\partial u_q}\left\{\frac{\partial U_{q,s}}{\partial x_1}+\sum_{p=1}^{n} U_{p,1}\frac{\partial U_{q,s}}{\partial u_p}\right\}.\end{aligned}$$

But by the differential equation (3) we have

$$\frac{\partial \xi_r}{\partial x_1}+\sum_{p=1}^{n} U_{p,1}\frac{\partial \xi_r}{\partial u_p}=0,$$

so that, differentiating for explicit occurrence of x_s, we have

$$\begin{aligned}\frac{\partial^2 \xi_r}{\partial x_1\partial x_s}+\sum_{p=1}^{n} U_{p,1}\frac{\partial^2 \xi_r}{\partial x_s\partial u_p}&=-\sum_{p=1}^{n}\frac{\partial U_{p,1}}{\partial x_s}\frac{\partial \xi_r}{\partial u_p}\\&=-\sum_{q=1}^{n}\frac{\partial U_{q,1}}{\partial x_s}\frac{\partial \xi_r}{\partial u_q},\end{aligned}$$

which transforms the first line of the expression for $\frac{\partial V_{rs}}{\partial x_1}$.

Similarly, differentiating the equation for explicit occurrence of u_q, we have

$$\frac{\partial^2\xi_r}{\partial u_q \partial x_1} + \sum_{p=1}^{n} U_{p,1} \frac{\partial^2 \xi_r}{\partial u_q \partial u_p} = - \sum_{p=1}^{n} \frac{\partial U_{p,1}}{\partial u_q} \frac{\partial \xi_r}{\partial u_p},$$

and therefore

$$\begin{aligned} \sum_{q=1}^{n} U_{qs} \left\{ \frac{\partial^2\xi_r}{\partial u_q \partial x_1} + \sum_{p=1}^{n} U_{p,1} \frac{\partial^2 \xi_r}{\partial u_q \partial u_p} \right\} &= - \sum_{q=1}^{n} \sum_{p=1}^{n} U_{q,s} \frac{\partial U_{p,1}}{\partial u_q} \frac{\partial \xi_r}{\partial u_p} \\ &= - \sum_{p=1}^{n} \sum_{q=1}^{n} U_{p,s} \frac{\partial U_{q,1}}{\partial u_p} \frac{\partial \xi_r}{\partial u_q}, \end{aligned}$$

which transforms the second line of the expression for $\dfrac{\partial V_{rs}}{\partial x_1}$. Hence, on substitution,

$$\begin{aligned} \frac{\partial V_{rs}}{\partial x_1} &= \sum_{q=1}^{n} \frac{\partial \xi_r}{\partial u_q} \left\{ \frac{\partial U_{q,s}}{\partial x_1} - \frac{\partial U_{q,1}}{\partial x_s} + \sum_{p=1}^{n} \left(U_{p,1} \frac{\partial U_{q,s}}{\partial u_p} - U_{p,s} \frac{\partial U_{q,1}}{\partial u_p} \right) \right\} \\ &= 0, \end{aligned}$$

because the coefficient of every one of the terms $\dfrac{\partial \xi_r}{\partial u_q}$ $(q = 1, 2, \ldots, n)$ vanishes by the conditions (2). Hence V_{rs} is explicitly independent of x_1; and therefore the system of equations (I) has been transformed to the system

$$dv_r = \sum_{s=2}^{m} V_{rs} dx_s \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{I})',$$

in which x_1 no longer occurs explicitly.

29. Some of the conditions (2) have been used in making this elimination of x_1 possible: viz. those of (2) which hold for

(i) each of the values $1, 2, \ldots, n$ of r; simultaneously with

(ii) all the $m - 1$ combinations of s with 1.

Hence $n(m-1)$ conditions have been used; and there thus remain $\frac{1}{2}nm(m-1) - n(m-1)$, that is, $\frac{1}{2}n(m-1)(m-2)$ conditions, as yet unused; they are those equations of (2) which hold for

(i) each of the values $1, 2, \ldots, n$ of r; simultaneously with

(ii) each of the $\frac{1}{2}(m-1)(m-2)$ combinations of s and t from the series $2, 3, \ldots, m$.

These remaining conditions can be transformed. We have

$$V_{r,t} = \frac{\partial \xi_r}{\partial x_t} + \sum_{q=1}^{n} U_{q,t} \frac{\partial \xi_r}{\partial u_q};$$

and hence

$$\begin{aligned}
\frac{\partial V_{r,t}}{\partial x_s} + \sum_{p=2}^{n} \frac{\partial V_{r,t}}{\partial v_p} \frac{\partial v_p}{\partial x_s} &= \frac{\partial^2 \xi_r}{\partial x_s \partial x_t} + \sum_{l=1}^{n} \frac{\partial^2 \xi_r}{\partial x_t \partial u_l} U_{l,s} \\
&\quad + \sum_{q=1}^{n} U_{q,t} \left\{ \frac{\partial^2 \xi_r}{\partial u_q \partial x_s} + \sum_{l=1}^{n} \frac{\partial^2 \xi_r}{\partial u_q \partial u_l} U_{l,s} \right\} \\
&\quad + \sum_{q=1}^{n} \frac{\partial \xi_r}{\partial u_q} \left\{ \frac{\partial U_{q,t}}{\partial x_s} + \sum_{l=1}^{n} \frac{\partial U_{q,t}}{\partial u_l} U_{l,s} \right\} \\
&= \frac{\partial^2 \xi_r}{\partial x_s \partial x_t} + \sum_{q=1}^{n} \left(U_{q,s} \frac{\partial^2 \xi_r}{\partial u_q \partial x_t} + U_{q,t} \frac{\partial^2 \xi_r}{\partial u_q \partial x_s} \right) \\
&\quad + \sum_{q=1}^{n} \sum_{p=1}^{n} U_{q,t} U_{p,s} \frac{\partial^2 \xi_r}{\partial u_p \partial u_q} \\
&\quad + \sum_{q=1}^{n} \frac{\partial \xi_r}{\partial u_q} \left\{ \frac{\partial U_{q,t}}{\partial x_s} + \sum_{p=1}^{n} \frac{\partial U_{q,t}}{\partial u_p} U_{p,s} \right\}.
\end{aligned}$$

Also, since $$v_p = \xi_p,$$

we have $$\frac{\partial v_p}{\partial x_s} = \frac{\partial \xi_p}{\partial x_s} + \sum_{q=1}^{n} \frac{\partial \xi_p}{\partial u_q} U_{q,s} = V_{p,s}$$

by (4): a result which might also have been inferred from (I)′. Hence

$$\begin{aligned}
\frac{\partial V_{r,t}}{\partial x_s} + \sum_{p=2}^{n} V_{p,s} \frac{\partial V_{r,t}}{\partial v_p} &= \frac{\partial^2 \xi_r}{\partial x_s \partial x_t} + \sum_{q=1}^{n} \left(U_{q,s} \frac{\partial^2 \xi_r}{\partial u_q \partial x_t} + U_{q,t} \frac{\partial^2 \xi_r}{\partial u_q \partial x_s} \right) \\
&\quad + \sum_{q=1}^{n} \sum_{p=1}^{n} U_{q,t} U_{p,s} \frac{\partial^2 \xi_r}{\partial u_p \partial u_q} + \sum_{q=1}^{n} \frac{\partial \xi_r}{\partial u_q} \left\{ \frac{\partial U_{q,t}}{\partial x_s} + \sum_{p=1}^{n} \frac{\partial U_{q,t}}{\partial u_p} U_{p,s} \right\}.
\end{aligned}$$

Interchanging s and t and subtracting the corresponding members of the two equations, we find

$$\begin{aligned}
&\frac{\partial V_{r,t}}{\partial x_s} - \frac{\partial V_{r,s}}{\partial x_t} + \sum_{p=2}^{n} \left(V_{p,s} \frac{\partial V_{r,t}}{\partial v_p} - V_{p,t} \frac{\partial V_{r,s}}{\partial v_p} \right) \\
&\qquad = \sum_{q=1}^{n} \frac{\partial \xi_r}{\partial u_q} \left\{ \frac{\partial U_{q,t}}{\partial x_s} - \frac{\partial U_{q,s}}{\partial x_t} + \sum_{p=1}^{n} \left(U_{p,s} \frac{\partial U_{q,t}}{\partial u_p} - U_{p,t} \frac{\partial U_{q,s}}{\partial u_p} \right) \right\},
\end{aligned}$$

the covariantive character of which equation is worthy of notice.

Since all the coefficients of terms $\frac{\partial \xi_r}{\partial u_q}$ on the right-hand side are zero, it follows that

$$\frac{\partial V_{r,t}}{\partial x_s} - \frac{\partial V_{r,s}}{\partial x_t} + \sum_{p=2}^{n} \left(V_{p,s} \frac{\partial V_{r,t}}{\partial v_p} - V_{p,t} \frac{\partial V_{r,s}}{\partial v_p} \right) = 0 \ldots (2)',$$

an equation which holds for

(i) each of the values $1, 2, \ldots, n$ of r;

(ii) each of the $\frac{1}{2}(m-1)(m-2)$ combinations of s and t from the series $2, 3, \ldots, m$.

Hence (2)′ implies

$$\tfrac{1}{2}n(m-1)(m-2)$$

conditions in all. Moreover, because the quantities ξ_r are independent quà functions of $u_1, \ldots, u_n$, so that their Jacobian does not vanish, the conditions (2)′ conversely lead to the vanishing of all the coefficients of the quantities $\frac{\partial \xi_r}{\partial u_q}$ in the previous equations. It therefore follows that the $\frac{1}{2}n(m-1)(m-2)$ conditions (2)′ are equivalent to and coextensive with the $\frac{1}{2}n(m-1)(m-2)$ conditions which survived after the transformation (I)′; and therefore these new conditions (2)′ may replace the old.

30. The new conditions (2)′ stand in the same relation to the new equations (I)′ as the original conditions (2) to the original equations (I). Hence if the system of equations (I)′, involving n dependent and $m-1$ independent variables, have its integral equivalent in the form of a system of n integral equations in virtue of the conditions (2)′, it follows that the system of equations (I) involving n dependent and m independent variables has its integral equivalent in the form of a system of n integral equations in virtue of the conditions (2).

Now for a single independent variable the conditions are evanescent; and it is known that the integral equivalent then consists of n equations. Hence, by induction, we have the following theorem:—

[Th] *In order that a system of n simultaneous linear differential equations*

$$du_r = \sum_{t=1}^{m} U_{r,t}\, dx_t,$$

where $r = 1, 2, \ldots, n$ and the coefficients $U_{r,t}$ are functions of all the variables x and u, may possess an integral equivalent in the form of a system of n integral equations, it is necessary that the conditions

$$\frac{\partial U_{r,t}}{\partial x_s} - \frac{\partial U_{r,s}}{\partial x_t} + \sum_{p=1}^{n} \left(U_{p,s} \frac{\partial U_{r,t}}{\partial u_p} - U_{p,t} \frac{\partial U_{r,s}}{\partial u_p} \right) = 0$$

should be satisfied for, (i) *each of the values* $1, 2, \ldots, n$ *of* r *and* (ii) *each of the* $\frac{1}{2}m(m-1)$ *possible combinations of* s *and* t *from the series* $1, 2, \ldots, m$; *and the identical satisfaction of these conditions is sufficient to ensure the existence of an integral equivalent of the stated form**.

31. The following method† of expressing the necessary conditions, which will (after the result just proved) be assumed sufficient, refers to the unsimplified or uncanonical form of the equations. Let the equations involve n variables and be k in number, where k is not greater than n: and suppose them to be

$$\left.\begin{array}{c} X_{11}\,dx_1 + \ldots\ldots + X_{1n}dx_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ X_{k1}\,dx_1 + \ldots\ldots + X_{kn}dx_n = 0 \end{array}\right\}.$$

Let the coefficients X be such that only m of these equations are independent; then all the remaining $k - m$ equations are linear combinations of those m, which may be taken to be the first m equations. We proceed to express the necessary (and sufficient) conditions that the m equations should be an exact system and have therefore m independent integrals.

If one of these supposed integrals be $f =$ constant, then the differential relation

$$\frac{\partial f}{\partial x_1}\,dx_1 + \ldots\ldots + \frac{\partial f}{\partial x_n}\,dx_n = 0$$

must be a linear combination of the m independent equations above; and therefore all determinants of the $(m+1)^{\text{th}}$ order of the system

$$\ldots\ldots \left\| \begin{array}{c} X_{11},\ \ldots\ldots\ldots,\ X_{1n} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ X_{m1},\ \ldots\ldots\ldots,\ X_{mn} \\ \dfrac{\partial f}{\partial x_1},\ \ldots\ldots\ldots,\ \dfrac{\partial f}{\partial x_n} \end{array} \right\| \ldots\ldots$$

* This result appears to have been given first by Deahna, *Crelle*, t. xx. (1840), pp. 340—349.

† Frobenius, *Crelle*, t. lxxxii. (1877), pp. 276 et seq.

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must vanish. Hence all systems of quantities ξ, which satisfy the equations

$$X_{\mu 1}\xi_1 + \ldots\ldots + X_{\mu n}\xi_n = 0$$

(for $\mu = 1, \ldots, m$), also satisfy the equation

$$\Xi(f) = \frac{\partial f}{\partial x_1}\xi_1 + \ldots\ldots + \frac{\partial f}{\partial x_n}\xi_n = 0,$$

which is thus a differential equation for the integral f. And, since each of the original equations can be expressed as a linear combination of the m independent equations, we may change the series $1, 2, \ldots, m$ as values for μ to the series $1, 2, \ldots, k$.

Since m of the equations connecting the n quantities ξ are independent, there are $n - m$ essentially distinct systems* of solutions, and therefore there are $n - m$ distinct differential equations of the form $\Xi f = 0$. Since there are m functions f and there are $n - m$ linear differential equations in the n variables, this system of partial differential equations is complete; and the conditions for this completeness are the conditions for the completeness of the independent equations of the original system.

Let $u_1, \ldots, u_n$; $v_1, \ldots, v_n$; be any two systems of solutions of the quantities ξ; and let

$$U(f) = 0, \qquad V(f) = 0$$

be the corresponding differential equations. Then the required conditions are that, for every possible pair, the equations

$$U\{V(f)\} = V\{U(f)\}$$

shall be satisfied either identically or in virtue of the $n - m$ differential equations $\Xi(f) = 0$. Now

* If $P_{\lambda 1}, \ldots, P_{\lambda n}$ (for $\lambda = 1, \ldots, n - m$) be $n - m$ new sets of quantities such that the determinant

$$\Delta = \begin{vmatrix} X_{11}, \ldots\ldots\ldots\ldots, X_{1n} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ X_{m1}, \ \ldots\ldots\ldots, X_{mn} \\ P_{11}, \ldots\ldots\ldots\ldots, P_{1n} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ P_{n-m,1}, \ \ldots\ldots, P_{n-m,n} \end{vmatrix}$$

does not vanish, then a system of solutions is given by

$$\xi_1 : \ldots\ldots : \xi_n = \frac{\partial \Delta}{\partial P_{\lambda,1}} : \ldots\ldots, \frac{\partial \Delta}{\partial P_{\lambda n}};$$

and the $n - m$ systems, corresponding to all the possible values of λ, are distinct.

$$U\{V(f)\} - V\{U(f)\} = \sum_{i=1}^{n} u_i \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} v_j \frac{\partial f}{\partial x_j}\right) - \sum_{j=1}^{n} v_j \frac{\partial}{\partial x_j}\left(\sum_{i=1}^{n} u_i \frac{\partial f}{\partial x_i}\right)$$
$$= \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \sum_{i=1}^{n} \left(u_i \frac{\partial v_j}{\partial x_i} - v_i \frac{\partial u_j}{\partial x_i}\right),$$

which must vanish. The system of partial differential equations being complete and each equation of the system being linear and homogeneous in the partial differential coefficients, it follows that there must exist some solution of the equations determining the quantities ξ such that

$$w_j = \sum_{i=1}^{n} \left(u_i \frac{\partial v_j}{\partial x_i} - v_i \frac{\partial u_j}{\partial x_i}\right)$$

for $j = 1, \ldots, n$; and the equations, which arise from this condition for all values of j and for all possible pairs of distinct sets of solutions, are sufficient to ensure the completeness of the system. Now by the definition of the quantities w_j we have for all the values of μ the relation

$$\sum_{j=1}^{n} X_{\mu j} w_j = 0,$$

and therefore for all the values $1, \ldots, k$ of μ the relations

$$\sum_{i=1}^{n} \sum_{j=1}^{n} X_{\mu j} u_i \frac{\partial v_j}{\partial x_i} = \sum_{i=1}^{n} \sum_{j=1}^{n} X_{\mu j} v_i \frac{\partial u_j}{\partial x_i},$$

though the equations in this set for $\mu = m+1, m+2, \ldots, k$ are superfluous. But also

$$\sum_{j=1}^{n} X_{\mu j} v_j = 0,$$

so that
$$\sum_{j=1}^{n} X_{\mu j} \frac{\partial v_j}{\partial x_i} = -\sum_{j=1}^{n} v_j \frac{\partial X_{\mu j}}{\partial x_i};$$

and similarly
$$\sum_{j=1}^{n} X_{\mu j} \frac{\partial u_j}{\partial x_i} = -\sum_{j=1}^{n} u_j \frac{\partial X_{\mu j}}{\partial x_i}.$$

Hence sufficient and necessary conditions are included in the set of equations

$$\sum_{i=1}^{n} \sum_{j=1}^{n} u_i v_j \frac{\partial X_{\mu j}}{\partial x_i} = \sum_{i=1}^{n} \sum_{j=1}^{n} u_j v_i \frac{\partial X_{\mu j}}{\partial x_i},$$

or finally, in the set

$$\sum_{i=1}^{n} \sum_{j=1}^{n} u_i v_j \left(\frac{\partial X_{\mu j}}{\partial x_i} - \frac{\partial X_{\mu i}}{\partial x_j}\right) = 0,$$

for the values 1, 2, ..., k of μ (some of which equations are superfluous because not independent of the others) and for all pairs of sets of distinct solutions u and v of the equations

$$X_{\mu 1}\xi_1 + \ldots\ldots + X_{\mu n}\xi_n = 0$$

for the values of 1, 2, ..., k of μ.

Ex. As an illustration of these results (for other details in regard to which the quoted memoir of Frobenius may be consulted), consider the system of equations

$$a_{i1}dx_1 + a_{i2}dx_2 + \ldots\ldots + a_{in}dx_n = 0 \quad (i = 1, \ldots\ldots, n),$$

so that k of the preceding investigation is n; and the quantities a_{ij} are supposed to be derived from n functions $P_1, \ldots, P_n$ by the equations

$$a_{ij} = \frac{\partial P_i}{\partial x_j} - \frac{\partial P_j}{\partial x_i};$$

and, as above, it will be assumed that minors of order m ($< n$) of the determinant

$$\begin{vmatrix} a_{11}, \ldots\ldots\ldots\ldots, a_{1n} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}, \ldots\ldots\ldots\ldots, a_{nn} \end{vmatrix}$$

do not all vanish, but that all minors of order greater than m do vanish.

In order that this system of m independent equations may be complete, the necessary and sufficient conditions are that the equations

$$\sum_{i=1}^{n}\sum_{j=1}^{n} u_i v_j \left(\frac{\partial a_{\mu j}}{\partial x_i} - \frac{\partial a_{\mu i}}{\partial x_j}\right) = 0$$

are satisfied for all values $\mu = 1, \ldots, n$ and for all sets of quantities which are subject to the equations (the independent equations being m in number),

$$\sum_{r=1}^{n} a_{\mu r}\xi_r = 0.$$

Since
$$\sum_{j=1}^{n} a_{ij}v_j = 0,$$

we have
$$\sum_{j=1}^{n} \frac{\partial a_{ij}}{\partial x_\mu} v_j + \sum_{j=1}^{n} a_{ij}\frac{\partial v_j}{\partial x_\mu} = 0;$$

and therefore, multiplying by u_i, adding, and remembering that for every value of j

$$\sum_{i=1}^{n} a_{ij}u_i = 0,$$

we have
$$\sum_{i=1}^{n}\sum_{j=1}^{n} u_i v_j \frac{\partial a_{ij}}{\partial x_\mu} = 0,$$

or since
$$\frac{\partial a_{ij}}{\partial x_\mu} + \frac{\partial a_{j\mu}}{\partial x_i} + \frac{\partial a_{\mu i}}{\partial x_j} = 0,$$

the conditions, above proved to be requisite, are all satisfied; and therefore the system is a complete system.

32. The integration of the equations, on the supposition that the conditions are satisfied, may be founded on the preceding investigation (§§ 27—30). One process follows it directly, and so is a generalisation of Euler's method; another, after the adoption of a modification, comes to be Natani's method.

For the former process, we begin by supposing all the variables constant except x_1 and integrate the equations

$$du_r = U_{r1} dx_1 \quad (r = 1, 2, \ldots, n)$$

in the forms $\qquad \xi_r = \text{constant} = v_r\,(x_2, \ldots, x_m),$

where v_r is unknown. When these values of ξ are substituted, the equations are of the form

$$dv_r = \sum_{t=2}^{m} V_{rt}\, dx_t,$$

where V_{rt} does not involve x_1. The similar transformation is applied to the new system in order to diminish the number of independent variables by unity; and so on, until the integral is obtained.

Ex. 1. To integrate

$$(v-u)\,du = (1-uy)\,dx + (1-ux)\,dy,$$

$$(v-u)\,dv = (vy-1)\,dx + (vx-1)\,dy.$$

The two necessary conditions are satisfied. Proceeding as indicated, we find two independent integrals of

$$(v-u)\,du = (1-uy)\,dx,$$

$$(v-u)\,dv = (vy-1)\,dx,$$

on the supposition that y is constant. We find them to be

$$u + v = xy + \theta,$$

$$uv = x \;\; + \phi,$$

where θ and ϕ, the constants of integration, are made functions of y. Then

$$d\theta = du + dv - xdy - ydx$$

$$= 0,$$

after substitution for du and dv from the original equations; and

$$d\phi = udv + vdu - dx$$

$$= dy,$$

after the same substitutions. Hence

$$\theta = a,$$

$$\phi = y + \beta,$$

where α and β are constants; and thus the integral equivalent of the original system is

$$\left.\begin{array}{l} u+v-xy=\alpha \\ uv-x-y=\beta \end{array}\right\}.$$

Evidently any function of $u+v-xy$ and $uv-x-y$ is a solution of the equations, that is, its variation is zero in virtue of the differential equations.

Ex. 2. The system

$$\frac{(u-v)(u-w)\,du}{(ux-1)(uy-1)(uz-1)}=\frac{dx}{ux-1}+\frac{dy}{uy-1}+\frac{dz}{uz-1},$$

$$\frac{(v-u)(v-w)\,dv}{(vx-1)(vy-1)(vz-1)}=\frac{dx}{vx-1}+\frac{dy}{vy-1}+\frac{dz}{vz-1},$$

$$\frac{(w-u)(w-v)\,dw}{(wx-1)(wy-1)(wz-1)}=\frac{dx}{wx-1}+\frac{dy}{wy-1}+\frac{dz}{wz-1},$$

may be similarly solved, with the result

$$\text{constant}=uvw-x-y-z,$$
$$\text{constant}=uv+vw+wu-xy-yz-zx,$$
$$\text{constant}=u+v+w-xyz.$$

33. The other method is that given by Natani*; it is the natural generalisation of his method which applies to the case of a single equation, and it is connected with the generalisation of Euler's method in the same way as his earlier method is connected (see Ex. 2, § 19) with Euler's method and may be justified by a similar argument.

Ex. 1. Sufficient illustration will be afforded as regards the general process by considering a set of equations

$$\left.\begin{array}{l} du=Udx+U'dy+U''dz \\ dv=Vdx+V'dy+V''dz \\ dw=Wdx+W'dy+W''dz \end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(a),$$

supposed to be integrable. The following is Natani's method: the possibility of its application at each successive step is implicitly proved in the earlier investigation.

The integrals of

$$du=Udx,\qquad dv=Vdx,\qquad dw=Wdx,$$

on the supposition of y and z constant, are of the form

$$\left.\begin{array}{l} \phi(u,v,w,x,y,z)=\text{constant}=\rho(y,z) \\ \psi(u,v,w,x,y,z)=\text{constant}=\sigma(y,z) \\ \chi(u,v,w,x,y,z)=\text{constant}=\tau(y,z) \end{array}\right\}\ldots\ldots\ldots\ldots\ldots(a)',$$

* *Crelle*, t. lviii., p. 303.

where the functions ρ, σ, τ are not derivable by this first integration, but the forms ϕ, ψ, χ are known. When, however, the values of ρ, σ, τ are known, then the foregoing equations constitute the integral equivalent of the system of differential equations.

Since the three functions on the right-hand side do not involve x, their value and form are not altered when any particular value, say zero, is assigned to x. Let u_1, v_1, w_1 be the corresponding values of u, v, w; then we have

$$\left.\begin{aligned}\phi(u, v, w, x, y, z) &= \phi(u_1, v_1, w_1, 0, y, z)\\ \psi(u, v, w, x, y, z) &= \psi(u_1, v_1, w_1, 0, y, z)\\ \chi(u, v, w, x, y, z) &= \chi(u_1, v_1, w_1, 0, y, z)\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(a)'';$$

and the equations determining u_1, v_1, w_1 are

$$\left.\begin{aligned}du_1 &= U_1'dy + U_1''dz\\ dv_1 &= V_1'dy + V_1''dz\\ dw_1 &= W_1'dy + W_1''dz\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(b),$$

where the coefficients are the values of the original coefficients after the substitutions $u=u_1$, $v=v_1$, $w=w_1$, $x=0$.

Let the integrals of

$$du_1 = U_1'dy, \qquad dv_1 = V_1'dy, \qquad dw_1 = W_1'dy,$$

on the supposition of z constant, be

$$\left.\begin{aligned}\xi(u_1, v_1, w_1, y, z) &= \text{constant} = \lambda(z)\\ \eta(u_1, v_1, w_1, y, z) &= \text{constant} = \mu(z)\\ \zeta(u_1, v_1, w_1, y, z) &= \text{constant} = \nu(z)\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(b)',$$

where the functions λ, μ, ν are not derivable by this integration, but the forms of ξ, η, ζ are known. Since those three functions do not involve y, they remain unaltered by the assignation of some special value, say zero, to y. Let u_2, v_2, w_2 be the corresponding values of u_1, v_1, w_1; then we have

$$\left.\begin{aligned}\xi(u_1, v_1, w_1, y, z) &= \xi(u_2, v_2, w_2, 0, z)\\ \eta(u_1, v_1, w_1, y, z) &= \eta(u_2, v_2, w_2, 0, z)\\ \zeta(u_1, v_1, w_1, y, z) &= \zeta(u_2, v_2, w_2, 0, z)\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(b)'';$$

and the equations determining u_2, v_2, w_2 are

$$du_2 = U_2''dz, \qquad dv_2 = V_2''dz, \qquad dw_2 = W_2''dz\ldots\ldots\ldots\ldots\ldots(c),$$

where the coefficients are the values of U_1'', V_1'', W_1'' after the substitutions $u_1=u_2$, $v_1=v_2$, $w_1=w_2$, $y=0$, i.e. are the values of the original coefficients U'', V'', W'' after the substitutions $u=u_2$, $v=v_2$, $w=w_2$, $x=0$, $y=0$.

The integrals of (c) are of the form

$$\left.\begin{aligned}\alpha(u_2, v_2, w_2, z) &= a\\ \beta(u_2, v_2, w_2, z) &= \beta\\ \gamma(u_2, v_2, w_2, z) &= \gamma\end{aligned}\right\},$$

where the quantities on the right-hand side are constants. From these equations we determine u_2, v_2, w_2 as functions of z; when they are substituted in $(b)''$, we have, by comparison with $(b)'$, the values of $\lambda(z)$, $\mu(z)$, $\nu(z)$.

We now use $(b)'$ to determine u_1, v_1, w_1 in terms of y and z; when their values are substituted in $(a)''$ we have, by comparison with $(a)'$, the values of $\rho(y, z)$, $\sigma(y, z)$, $\tau(y, z)$. *Then $(a)'$ gives the integral equivalent of the system (a).*

Applied to the particular example 1 of § 32, the first integration gives

$$u+v-xy=\text{function of } y=u_1+v_1,$$

$$uv-x=\text{function of } y=u_1v_1,$$

on putting $x=0$; and u_1 and v_1 are determined by

$$(v_1-u_1)\,du_1=\quad dy,$$

$$(v_1-u_1)\,dv_1=-dy,$$

so that

$$u_1+v_1=a,$$

$$u_1v_1-y=\beta.$$

Hence

$$u+v-xy=a,$$

$$uv-x\;=y+\beta,$$

the same solution as before.

Ex. 2. Integrate the system of equations

$$\left.\begin{aligned}
du_1&=u_1\frac{-2x_1+x_2+a}{(x_1-x_2)(x_1-a)}dx_1+u_2\frac{a-x_2}{(x_1-x_2)(x_1-a)}dx_2\\
du_2&=u_1\frac{a-x_1}{(x_2-x_1)(x_2-a)}dx_1+u_2\frac{-2x_2+x_1+a}{(x_2-x_1)(x_2-a)}dx_2\\
du_3&=u_1\frac{x_2-x_1}{(a-x_1)(a-x_2)}dx_1+u_2\frac{x_1-x_2}{(a-x_1)(a-x_2)}dx_2
\end{aligned}\right\}.$$

(Maximowitch.)

Ex. 3. Integrate the simultaneous system of total equations :—

$$dz_1=\frac{1}{(x_1-x_2)(x_1-x_3)}[z_1(-2x_1+x_2+x_3)\,dx_1+z_2(x_3-x_2)\,dx_2+z_3(x_2-x_3)\,dx_3]$$

$$dz_2=\frac{1}{(x_2-x_1)(x_2-x_3)}[z_1(x_3-x_1)\,dx_1+z_2(-2x_2+x_1+x_3)\,dx_2+z_3(x_1-x_3)\,dx_3]$$

$$dz_3=\frac{1}{(x_3-x_1)(x_3-x_2)}[z_1(x_2-x_1)\,dx_1+z_2(x_1-x_2)\,dx_2+z_3(-2x_3+x_1+x_2)\,dx_3].$$

(Maximowitch.)

Ex. 4. Integrate the (exact) equations :

$$\left.\begin{aligned}
du&=(u+x)\,dx+(v+y+1)\,dy+(w+z+1)\,dz\\
dv&=(v+y+1)\,dx+(w+z)\,dy+(u+x+1)\,dz\\
dw&=(w+z+1)\,dx+(u+x+1)\,dy+(v+y)\,dz
\end{aligned}\right\};$$

and also

$$\left.\begin{aligned}
du&=udx+vdy+wdz\\
dv&=vdx+wdy+udz\\
dw&=wdx+udy+vdz
\end{aligned}\right\};$$

(Stodockiewicz.)

Ex. 5. Integrate the simultaneous equations

$$\left.\begin{aligned} dy_1 &= (a_{11}y_1 + a_{12}y_2)\,dx_1 + (a_{21}y_1 + a_{22}y_2)\,dx_2 \\ dy_2 &= (b_{11}y_1 + b_{12}y_2)\,dx_1 + (b_{21}y_1 + b_{22}y_2)\,dx_2 \end{aligned}\right\},$$

where the quantities a and b are functions of x_1 and x_2, supposed to satisfy the conditions necessary for integration.

(Le Pont.)

34. A very remarkable development of Natani's method of integration has been made by Mayer*.

As indicated in § 33, Natani's method when applied to the general system of integrable equations (I) requires the integration of m different systems of n ordinary differential equations of the first order; and the successive systems are constructed by the adoption of special values for those of the variables whose variations have ceased to occur. Mayer's method requires the integration of *only one* system of n ordinary differential equations of the first order.

In § 33 it was assumed (for purposes of illustration) that zero was a suitable value for each of the variables, when once its variation ceased to occur: suppose more generally that for the independent variables in the equations

$$du_r = \sum_{t=1}^{m} U_{r,t}\,dx_t \quad \text{.........................(I)}$$

the values $x_1 = \alpha_1$, $x_2 = \alpha_2$, $x_3 = \alpha_3$, ..., are assigned after the integration of each of the successive systems of ordinary equations.

The independent variables are changed to $y_1, \ldots, y_m$, any set of m independent quantities, by the substitutions

$$x_t = \alpha_t + (y_1 - \theta) f_t,$$

where $f_1, \ldots, f_m$ are m independent functions of the new variables y, and θ is a constant. When substitution takes place in the system (I), it takes the form

$$du_r = \sum_{s=1}^{m} Y_{rs} dy_s \quad \text{.........................(I)}',$$

where
$$Y_{r1} = \sum_{t=1}^{m} U_{rt} \left\{ f_t + (y_1 - \theta) \frac{\partial f_t}{\partial y_1} \right\},$$

$$Y_{rs} = (y_1 - \theta) \sum_{t=1}^{m} U_{rt} \frac{\partial f_t}{\partial y_s}.$$

* *Math. Ann.*, t. v. (1872), pp. 449—470, "Ueber simultane, totale und partielle Differentialgleichungen"; especially § 3.

It is natural to expect that the system (I)′, which is merely a transformation of (I), will satisfy all the conditions of integrability: and it is easy to prove that

$$\frac{\partial Y_{rt}}{\partial y_s} - \frac{\partial Y_{rs}}{\partial y_t} + \sum_{p=1}^{n}\left(Y_{ps}\frac{\partial Y_{rt}}{\partial u_p} - Y_{pt}\frac{\partial Y_{rs}}{\partial u_p}\right)$$
$$= \sum_{\lambda=1}^{m}\sum_{\mu=1}^{m}\frac{\partial x_\lambda}{\partial y_s}\frac{\partial x_\mu}{\partial y_t}\left[\frac{\partial U_{r\mu}}{\partial x_\lambda} - \frac{\partial U_{r\lambda}}{\partial x_\mu} + \sum_{p=1}^{n}\left(U_{p\lambda}\frac{\partial U_{r\mu}}{\partial u_p} - U_{p\mu}\frac{\partial U_{r\lambda}}{\partial u_p}\right)\right]$$
$$= 0,$$

because the conditions of integrability of (I) are supposed to be satisfied.

Proceeding with the transformed system (I)′ as in Natani's method, we first integrate the system of n differential equations

$$du_s = Y_{s1}dy_1 \quad (s = 1, 2, \ldots\ldots, n) \ldots\ldots\ldots\ldots\ldots (\text{A})$$

on the supposition of the invariability of $y_2, \ldots, y_m$. Their n integrals are of the form

$$\phi_r(u_1, \ldots\ldots, u_n, y_1, \ldots\ldots, y_m) = \lambda_r,$$

where λ_r, a constant of integration, is independent of y_1 and may be a function of $y_2, \ldots, y_m$. The value of λ_r is thus not altered when we assign any special value to y_1. Let the value θ be assigned to y_1 and suppose that the altered values of $u_1, \ldots, u_n$ are $u_1', \ldots, u_n'$; then we have

$$\phi_r(u_1', \ldots\ldots, u_n', \theta, y_2, \ldots\ldots, y_m) = \lambda_r,$$

so that the system of n integrals is of the form

$$\phi_r(u_1, \ldots\ldots, u_n, y_1, y_2, \ldots\ldots, y_m) = \phi_r(u_1', \ldots\ldots, u_n', \theta, y_2, \ldots\ldots, y_m).$$

The quantities u' must now be obtained; they are, as in § 33, determined by n equations

$$du_r' = \sum_{s=2}^{m} Y'_{rs}dy_s,$$

where Y'_{rs} is the value of Y_{rs} after the substitutions $u_t = u_t'$, $y_1 = \theta$ have been made. But these substitutions make Y_{rs} vanish, so that all the coefficients Y'_{rs} are zero: hence the equations determining the quantities u' are

$$du'_r = 0,$$

that is, *all the quantities u' are constants.* Hence the system of integrals of the transformed system (I)′ is

$$\phi_r(u_1, \ldots, u_n, y_1, y_2, \ldots, y_m) = \phi_r(c_1, \ldots, c_n, \theta, y_2, \ldots, y_m) \ldots \text{(B)},$$

the functions ϕ_r being determined by the integration of the equations (A), the only integration now necessary.

In order to obtain a system of integrals of the system (I) we merely have to eliminate the quantities y from the equations (B) by the relations of transformation

$$x_t = \alpha_t + (y_1 - \theta) f_t;$$

there are sufficient arbitrary constants to constitute the result a system of integrals of the equations (I).

35. To render the integration of the equations (A) as easy as possible, it is natural to adopt the simplest relations of transformation consistent with generality. These appear to be

$$x_1 = y_1,$$

and
$$x_r = \alpha_r + (y_1 - \alpha_1) y_r,$$

for $r = 2, 3, \ldots, m$; for these relations, the coefficients Y are

$$Y_{r1} = U_{r1} + \sum_{t=2}^{m} y_t U_{rt},$$

$$Y_{ri} = (y_1 - \alpha_1) U_{ri} \quad (i = 2, \ldots\ldots, m).$$

There is one simple property of the system of integrals which exists in some cases. If it be possible to eliminate the variables y from the function ϕ_r and obtain a function

$$\psi_r(u_1, \ldots\ldots, u_n, x_1, \ldots\ldots, x_m),$$

which is not indeterminate for values of x corresponding to $y_1 = \theta$, then the integral takes a simple form. The value of x_t which corresponds to $y_1 = \theta$ is α_t; hence the determinate value of ψ_r for $y_1 = \theta$, $u_1 = c_1, \ldots, u_n = c_n$ is

$$\psi_r(c_1, \ldots\ldots, c_n, \alpha_1, \ldots\ldots, \alpha_m),$$

which is thus the value of $\phi_r(c_1, \ldots, c_n, \theta, y_2, \ldots, y_m)$. Hence it follows that *when the value of the right-hand side of the integral* (B) *takes the form*

$$\psi_r(c_1, \ldots\ldots, c_n, \alpha_1, \ldots\ldots, \alpha_m)$$

a pure constant, the integral itself is given by

$$\psi_r(u_1, \ldots\ldots, u_n, x_1, \ldots\ldots, x_m) = \psi_r(c_1, \ldots\ldots, c_n, \alpha_1, \ldots\ldots, \alpha_m).$$

But it must be understood that this result in the present form is strictly limited to all integrals (B), which are such that the substitutions $y_1=\theta$, $u=c$ reduce the function ϕ_r to a constant. We give here one example in which this occurs; another will be given later in which the property does not hold.

Ex. As an illustration of Mayer's theorem, consider the equations

$$\left.\begin{aligned}(vy-ux)\,du&=(u^2+y^2)\,dx+(uv+xy)\,dy\\(ux-vy)\,dv&=(uv+xy)\,dx+(v^2+x^2)\,dy\end{aligned}\right\},$$

which satisfy the conditions of integrability. In accordance with Mayer's result connected with the simplest substitution, we leave x untransformed and take

$$y=\beta+(x-a)\,z;$$

then the set of subsidiary equations (A) is

$$\left.\begin{aligned}[v\{\beta+(x-a)\,z\}-ux]\,du&=\{u\,(u+vz)+(\beta-az+xz)\,(\beta-az+2xz)\}\,dx\\-[v\{\beta+(x-a)\,z\}-ux]\,dv&=\{v\,(u+vz)+(\beta-az+2xz)\,x\}\,dx\end{aligned}\right\}.$$

Two integrals of these are easily obtained (z is invariable) by taking them in the equivalent forms

$$vdu+udv=(\beta-az+2xz)\,dx,$$

$$xdu+(\beta-az+xz)\,dv=-(u+vz)\,dx.$$

The former gives

$$uv-(\beta-az)\,x-x^2z=c_1c_2-a\beta,$$

on taking $x=a$; hence we infer that the integral is

$$uv-xy=c_1c_2-a\beta=\text{constant}.$$

The latter gives

$$xu+(\beta-az)\,v+zxv=ac_1+\beta c_2,$$

on taking $x=a$ and of course $u=c_1$, $v=c_2$ as before; hence we infer that the integral is

$$xu+yv=ac_1+\beta c_2=\text{constant}.$$

Hence the integrals are

$$\left.\begin{aligned}uv-xy&=\text{constant}\\xu+yv&=\text{constant}\end{aligned}\right\}.$$

36. The conditions, necessary and sufficient to ensure that the equations

$$\left.\begin{aligned}du_1&=X_1dx+Y_1dy\\du_2&=X_2dx+Y_2dy\end{aligned}\right\}$$

can be satisfied by a system of two integral equations are (by § 30)

$$\left.\begin{aligned}\frac{\partial X_1}{\partial y}-\frac{\partial Y_1}{\partial x}+Y_1\frac{\partial X_1}{\partial u_1}-X_1\frac{\partial Y_1}{\partial u_1}+Y_2\frac{\partial X_1}{\partial u_2}-X_2\frac{\partial Y_1}{\partial u_2}&=0\\\frac{\partial X_2}{\partial y}-\frac{\partial Y_2}{\partial x}+Y_1\frac{\partial X_2}{\partial u_1}-X_1\frac{\partial Y_2}{\partial u_1}+Y_2\frac{\partial X_2}{\partial u_2}-X_2\frac{\partial Y_2}{\partial u_2}&=0\end{aligned}\right\}.$$

It is interesting to see how these conditions arise when a set of integrating factors (§ 24) is introduced; the process leads, in this special case, to a method of solution distinct from those which have preceded.

Let a set of integrating factors be λ and μ, so that

$$\lambda(-du_1 + X_1 dx + Y_1 dy) + \mu(-du_2 + X_2 dx + Y_2 dy) = d\phi;$$

then

$$\frac{\partial \phi}{\partial u_1} = -\lambda,$$

$$\frac{\partial \phi}{\partial u_2} = -\mu,$$

$$\frac{\partial \phi}{\partial x} = \lambda X_1 + \mu X_2,$$

$$\frac{\partial \phi}{\partial y} = \lambda Y_1 + \mu Y_2.$$

Hence we have the conditions

$$0 = \frac{\partial \lambda}{\partial u_2} - \frac{\partial \mu}{\partial u_1},$$

$$0 = \frac{\partial \lambda}{\partial x} + \frac{\partial}{\partial u_1}(\lambda X_1 + \mu X_2),$$

$$0 = \frac{\partial \lambda}{\partial y} + \frac{\partial}{\partial u_1}(\lambda Y_1 + \mu Y_2),$$

$$0 = \frac{\partial \mu}{\partial x} + \frac{\partial}{\partial u_2}(\lambda X_1 + \mu X_2),$$

$$0 = \frac{\partial \mu}{\partial y} + \frac{\partial}{\partial u_2}(\lambda Y_1 + \mu Y_2),$$

$$0 = \frac{\partial}{\partial y}(\lambda X_1 + \mu X_2) - \frac{\partial}{\partial x}(\lambda Y_1 + \mu Y_2).$$

From the first three of these we have

$$\left.\begin{aligned} \frac{\partial \lambda}{\partial x} + X_1 \frac{\partial \lambda}{\partial u_1} + X_2 \frac{\partial \lambda}{\partial u_2} &= -\lambda \frac{\partial X_1}{\partial u_1} - \mu \frac{\partial X_2}{\partial u_1} \\ \frac{\partial \lambda}{\partial y} + Y_1 \frac{\partial \lambda}{\partial u_1} + Y_2 \frac{\partial \lambda}{\partial u_2} &= -\lambda \frac{\partial Y_1}{\partial u_1} - \mu \frac{\partial Y_2}{\partial u_1} \end{aligned}\right\};$$

and from the last three it proves to be possible to eliminate $\frac{\partial \mu}{\partial x}, \frac{\partial \mu}{\partial y}, \frac{\partial \mu}{\partial u_2}$ with the result

$$X_1\left(\frac{\partial \lambda}{\partial y}+Y_2\frac{\partial \lambda}{\partial u_2}\right)-Y_1\left(\frac{\partial \lambda}{\partial x}+X_2\frac{\partial \lambda}{\partial u_2}\right)$$
$$=\lambda\left(\frac{\partial Y_1}{\partial x}-\frac{\partial X_1}{\partial y}+X_2\frac{\partial Y_1}{\partial u_2}-Y_2\frac{\partial X_1}{\partial u_2}\right)+\mu\left(\frac{\partial Y_2}{\partial x}-\frac{\partial X_2}{\partial y}+X_2\frac{\partial Y_2}{\partial u_2}-Y_2\frac{\partial X_2}{\partial u_2}\right).$$

When this is combined with the last two equations, we find

$$0=\lambda\left(\frac{\partial Y_1}{\partial x}-\frac{\partial X_1}{\partial y}+X_2\frac{\partial Y_1}{\partial u_2}-Y_2\frac{\partial X_1}{\partial u_2}+X_1\frac{\partial Y_1}{\partial u_1}-Y_1\frac{\partial X_1}{\partial u_1}\right)$$
$$+\mu\left(\frac{\partial Y_2}{\partial x}-\frac{\partial X_2}{\partial y}+X_2\frac{\partial Y_2}{\partial u_2}-Y_2\frac{\partial X_2}{\partial u_2}+X_1\frac{\partial Y_2}{\partial u_1}-Y_1\frac{\partial X_2}{\partial u_1}\right).$$

Now the ratio of $\lambda:\mu$ cannot be a quantity which is the same for all solutions; for otherwise

$$\frac{\dfrac{\partial \phi}{\partial u_2}}{\dfrac{\partial \phi}{\partial u_1}}=\text{this unchanging ratio}=\frac{\dfrac{\partial \psi}{\partial u_2}}{\dfrac{\partial \psi}{\partial u_1}};$$

that is, the Jacobian of ϕ and ψ would vanish and every integral would then be expressible in terms of only one so far as the dependent variables can occur. Hence the preceding equation for $\lambda:\mu$ is not determinate; and therefore the coefficients of λ and μ must vanish. These are the two conditions that the given system should have its integral equivalent composed of two equations.

Further the only independent equations which can be obtained free from the differential coefficients of μ are the pair

$$\frac{\partial \lambda}{\partial x}+X_1\frac{\partial \lambda}{\partial u_1}+X_2\frac{\partial \lambda}{\partial u_2}=-\lambda\frac{\partial X_1}{\partial u_1}-\mu\frac{\partial X_2}{\partial u_1},$$
$$\frac{\partial \lambda}{\partial y}+Y_1\frac{\partial \lambda}{\partial u_1}+Y_2\frac{\partial \lambda}{\partial u_2}=-\lambda\frac{\partial Y_1}{\partial u_1}-\mu\frac{\partial Y_2}{\partial u_1}.$$

The elimination of μ leads to a linear partial differential equation for λ; when λ is known, then either of the equations will determine μ.

There is, of course, a similar result so far as regards μ, viz.

the only independent equations which can be obtained free from the differential coefficients of λ are the pair

$$\frac{\partial \mu}{\partial x}+X_1\frac{\partial \mu}{\partial u_1}+X_2\frac{\partial \mu}{\partial u_2}=-\lambda\frac{\partial X_1}{\partial u_2}-\mu\frac{\partial X_2}{\partial u_2},$$

$$\frac{\partial \mu}{\partial y}+Y_1\frac{\partial \mu}{\partial u_1}+Y_2\frac{\partial \mu}{\partial u_2}=-\lambda\frac{\partial Y_1}{\partial u_2}-\mu\frac{\partial Y_2}{\partial u_2};$$

and a similar inference can be drawn from these.

(An apparent case of doubt sometimes arises if, for instance, in the former pair both the quantities $\dfrac{\partial X_2}{\partial u_1}$, $\dfrac{\partial Y_2}{\partial u_1}$ vanish. This implies that the equation

$$du_2=X_2dx+Y_2dy,$$

contains only three variables x, y, u_2: and the condition of integrability is satisfied, so that the equation can be integrated. And then without any further consideration of the equations for λ, the value of u_2 obtained from the integral can be substituted in the first equation

$$du_1=X_1dx+Y_1dy,$$

which will then be found to satisfy the condition of integrability and is an equation in three variables.)

37. The differential equation for λ is

$$\frac{\partial\lambda}{\partial x}\frac{\partial Y_2}{\partial u_1}-\frac{\partial\lambda}{\partial y}\frac{\partial X_2}{\partial u_1}+\frac{\partial\lambda}{\partial u_1}\left(X_1\frac{\partial Y_2}{\partial u_1}-Y_1\frac{\partial X_2}{\partial u_1}\right)+\frac{\partial\lambda}{\partial u_2}\left(X_2\frac{\partial Y_2}{\partial u_1}-Y_2\frac{\partial X_2}{\partial u_1}\right)$$
$$=\lambda\left(\frac{\partial X_2}{\partial u_1}\frac{\partial Y_1}{\partial u_1}-\frac{\partial X_1}{\partial u_1}\frac{\partial Y_2}{\partial u_1}\right);$$

and the subsidiary equations for its complete integral are

$$\frac{dx}{\dfrac{\partial Y_2}{\partial u_1}}=\frac{dy}{-\dfrac{\partial X_2}{\partial u_1}}=\frac{du_1}{X_1\dfrac{\partial Y_2}{\partial u_1}-Y_1\dfrac{\partial X_2}{\partial u_1}}=\frac{du_2}{X_2\dfrac{\partial Y_2}{\partial u_1}-Y_2\dfrac{\partial X_2}{\partial u_1}}$$
$$=\frac{d\log\lambda}{\dfrac{\partial X_2}{\partial u_1}\dfrac{\partial Y_1}{\partial u_1}-\dfrac{\partial X_1}{\partial u_1}\dfrac{\partial Y_2}{\partial u_1}}\quad\text{.....................(A).}$$

The complete integral is not necessary for our purpose. One value of λ determines one value of μ; when these are combined, they lead to an integral equation derivable from the given system. Similarly a second value of λ will lead to a second integral equation; and in terms of two integrals all the integrals

of the given system can be expressed. Hence *it is sufficient to have two independent values of* λ *which satisfy its equation;* and the simpler they are, the simpler in general will be the integration of the resulting integrable equation.

Ex. A sufficient indication of the method will be given, if we once more consider the equations

$$(u_2-u_1)\,du_1=(1-u_1y)\,dx+(1-u_1x)\,dy,$$
$$(u_2-u_1)\,du_2=(u_2y-1)\,dx+(u_2x-1)\,dy.$$

We have
$$X_1=\frac{1-u_1y}{u_2-u_1},\qquad Y_1=\frac{1-u_1x}{u_2-u_1},$$
$$X_2=\frac{u_2y-1}{u_2-u_1},\qquad Y_2=\frac{u_2x-1}{u_2-u_1}.$$

Substituting in the subsidiary equations (A), we find they take the form

$$\frac{dx}{u_2x-1}=\frac{dy}{1-u_2y}=\frac{du_1}{x-y}=\frac{du_2}{0}=\frac{d\,.\log\lambda}{0}.$$

As two solutions of the equation determining λ are sufficient, we take these in the form

$$\lambda=\text{constant}=1,$$
$$\lambda=u_2,$$

which are the simplest derivable from the foregoing set. When we substitute these in turn in

$$\mu\frac{\partial X_2}{\partial u_1}=-\lambda\frac{\partial X_1}{\partial u_1}-\frac{\partial\lambda}{\partial x}-X_1\frac{\partial\lambda}{\partial u_1}-X_2\frac{\partial\lambda}{\partial u_2},$$

we find for the respective values

$$\mu=1,$$
$$\mu=u_1,$$

so that $\left.\begin{matrix}\lambda=1\\ \mu=1\end{matrix}\right\}$, $\left.\begin{matrix}\lambda=u_2\\ \mu=u_1\end{matrix}\right\}$, are two sets of integrating factors; and so, taking

$$\lambda\,(du_1-X_1dx-Y_1dy)+\mu\,(du_2-X_2dx-Y_2dy)=d\phi,$$

for the two cases we have

$$\left.\begin{matrix}u_1+u_2-xy=\phi_1\\ u_1u_2-x-y=\phi_2\end{matrix}\right\}.$$

Only two integrals of the subsidiary system (A) have been obtained; two others (which, with the two $\lambda_1=1$, $\mu_1=1$ and $\lambda_2=u_2$, $\mu_2=u_1$, make up a complete system of independent integrals for λ and of independent derived integrals for μ) are

$$\lambda_3=u_1-xy,\qquad \lambda_4=u_1u_2-x-y,$$

with the derived values

$$\mu_3=u_2-xy,\qquad \mu_4=u_1u_2-x-y.$$

Now the investigation in § 24 shews that, if λ_1, μ_1; λ_2, μ_2; λ', μ' be sets of integrating factors, then

$$\frac{\lambda_2\mu' - \mu_2\lambda'}{\lambda_2\mu_1 - \mu_2\lambda_1}, \qquad \frac{\lambda_1\mu' - \mu_1\lambda'}{\lambda_2\mu_1 - \mu_1\lambda_2}$$

are solutions when they are independent of one another and are not mere constants. When $\mu' = \mu_3$, $\lambda' = \lambda_3$, these quantities become

$$u_1 + u_2 - xy, \ 1$$

respectively, so that $u_1 + u_2 - xy$ is the only solution thence derivable; and when $\mu' = \mu_4$, $\lambda' = \lambda_4$, they become

$$u_1 u_2 - x - y, \ 0$$

respectively, so that $u_1 u_2 - x - y$ is the only solution thence derivable. But, since $u_1 + u_2 - xy$ and $u_1 u_2 - x - y$ are functionally independent of one another, it follows from the general theory that

$$u_1 + u_2 - xy = a,$$

$$u_1 u_2 - x - y = \beta,$$

constitute a solution of the system of two equations.

38. There is also another method of obtaining the integrals, which is founded on the results of the general investigation proving the sufficiency of the necessary conditions. The integrals are considered as the common solutions of simultaneous partial differential equations*; and it is easy to see that the process of successive integration is only a modification of the generalisation (§ 32) of Euler's method.

Every solution ϕ of the system of n equations

$$du_s = \sum_{t=1}^{m} U_{s,t}\, dx_t \text{ ....(I)}$$

for the values $s = 1, 2, \ldots, n$, has been shewn (§ 22) to satisfy the m partial differential equations

$$\Delta_t \phi = \frac{\partial \phi}{\partial x_t} + \sum_{s=1}^{n} U_{s,t} \frac{\partial \phi}{\partial u_s} = 0 \text{ ....(II)}$$

for the values $t = 1, 2, \ldots, m$.

The Jacobian conditions

$$(\Delta_r, \Delta_t) = 0$$

for the coexistence of the system (II) and the possession of common solutions are

* Boole, *Phil. Trans.* 1862, pp. 437—454: Mayer, § 12 of the memoir cited in § 34 above.

$$\sum_{s=1}^{n} \frac{\partial \phi}{\partial u_s} \left\{ \frac{\partial U_{s,t}}{\partial x_r} - \frac{\partial U_{s,r}}{\partial x_t} + \sum_{p=1}^{n} \left(U_{p,r} \frac{\partial U_{s,t}}{\partial u_p} - U_{p,t} \frac{\partial U_{s,r}}{\partial u_p} \right) \right\} = 0.$$

But the coefficient of every term $\dfrac{\partial \phi}{\partial u_s}$ vanishes, on account of the equations (2) of condition (§ 26) among the coefficients, which are assumed to be satisfied; and therefore the Jacobian conditions are all satisfied. Hence the system (II) is of the type, which is often called a *complete system*.

We now proceed to prove that the system (II) has, on these suppositions, n independent integrals, where n is the total number of variables in the system less the number of equations in the system.

Taking then the first equation of (II), viz.

$$\Delta_1 \phi = \frac{\partial \phi}{\partial x_1} + U_{11} \frac{\partial \phi}{\partial u_1} + U_{21} \frac{\partial \phi}{\partial u_2} + \ldots\ldots + U_{n1} \frac{\partial \phi}{\partial u_n} = 0,$$

the system of equations subsidiary to the derivation of the most general solution is

$$\frac{dx_1}{1} = \frac{dx_2}{0} = \ldots\ldots = \frac{dx_m}{0} = \frac{du_1}{U_{11}} = \frac{du_2}{U_{21}} = \ldots\ldots = \frac{du_n}{U_{n1}}.$$

Let the necessary $n + m - 1$ integrals of this system be

$$x_2, x_3, \ldots\ldots, x_m, \xi_1, \xi_2, \ldots\ldots, \xi_n.$$

Then every solution of $\Delta_1 \phi = 0$ is expressible in terms of these quantities; and in order to have the simultaneous solutions of (II), it is sufficient to find what functional combinations of them will satisfy the remaining equations $\Delta_2 \phi = 0, \ldots, \Delta_m \phi = 0$.

Now if

$$\phi = f(x_2, \ldots\ldots, x_m, \xi_1, \ldots\ldots, \xi_n)$$

satisfy $\Delta_t \phi = 0$, we have

$$0 = \frac{\partial f}{\partial x_t} + \sum_{r=1}^{n} \frac{\partial f}{\partial \xi_r} \left\{ \frac{\partial \xi_r}{\partial x_t} + \sum_{s=1}^{n} U_{s,t} \frac{\partial \xi_r}{\partial u_s} \right\}$$

$$= \frac{\partial f}{\partial x_t} + \sum_{r=1}^{n} V_{r,t} \frac{\partial f}{\partial \xi_r} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{II})',$$

where $V_{r,t}$, the same function as before (§ 28), is a function of $x_2, \ldots, x_m, \xi_1, \ldots, \xi_n$ only. This new system (II)′ is a complete

system on account of the conditions satisfied by the coefficients $V_{r,t}$; and the simultaneous solutions of (II)′ are solutions of (II).

We thus have, instead of the complete system of m linear equations involving $m+n$ variables, a new complete system of $m-1$ linear equations involving $m+n-1$ variables; and all the solutions of the second system are solutions of the first, and conversely.

The new system is treated in the same way and is replaced by another complete system of $m-2$ linear equations involving $m+n-2$ variables: and all the solutions of the latter are solutions of the former and therefore also of (II), and conversely.

Proceeding in this way we find that all the solutions of (II) are solutions of a system of one equation involving $n+1$ variables. But it is known that such an equation has n independent solutions and that every solution can be expressed in terms of those n solutions; hence *the system* (II) *has n independent solutions and every common solution can be expressed in terms of them.*

39. In order, however, to render the method appropriate to the solution of (I), it is necessary to prove the following proposition, which is the converse of that at the beginning of § 22.

A set of n independent solutions of (II) *constitutes a system of integral equations equivalent to* (I).

Let $\phi_1, \phi_2, \ldots, \phi_n$ be the n independent solutions of (II); then the equations

$$\phi_1 = a_1,\ \phi_2 = a_2,\ \ldots\ldots,\ \phi_n = a_n$$

give
$$\sum_{t=1}^{m} \frac{\partial \phi_r}{\partial x_t} dx_t + \sum_{s=1}^{n} \frac{\partial \phi_r}{\partial u_s} du_s = 0,$$

for $r = 1, 2, \ldots, n$. But by the equations (II) we have

$$\frac{\partial \phi_r}{\partial x_t} = -\sum_{s=1}^{n} \frac{\partial \phi_r}{\partial u_s} U_{s,t},$$

so that, substituting, we have

$$\sum_{s=1}^{n} \frac{\partial \phi_r}{\partial u_s} \left\{ du_s - \sum_{t=1}^{m} U_{s,t}\, dx_t \right\} = 0.$$

This is a system of n equations, for $r = 1, 2, \ldots, n$. The system is

linear and homogeneous in the n quantities

$$du_s - \sum_{t=1}^{m} U_{s,t}\, dx_t;$$

and the determinant of the coefficients of those quantities is

$$\frac{\partial(\phi_1, \phi_2, \ldots\ldots, \phi_n)}{\partial(u_1, u_2, \ldots\ldots, u_n)},$$

which does not vanish, because the quantities ϕ are independent of one another. It follows therefore that

$$du_s - \sum_{t=1}^{m} U_{s,t}\, dx_t = 0$$

for $s = 1, 2, \ldots, n$; this is the system (I) and it holds in virtue of the equations $\phi_r = a_r$. The proposition is therefore proved.

Hence to obtain an integral equivalent of (I) it is sufficient to obtain any n independent common integrals of the system (II).

Ex. To solve the equations

$$\left.\begin{aligned} (1+xy)\, du &= (v+2x+xy)\, dx + (2y+x^2-xu)\, dy \\ (1+xy)\, dv &= (y-vy-2xy)\, dx + (x-u-2y^2)\, dy \end{aligned}\right\},$$

the necessary conditions for the integration of which are satisfied.

If $\phi(x, y, u, v)$ be a solution, the partial equations which determine ϕ are

$$(1+xy)\frac{\partial\phi}{\partial x} + (v+2x+xy)\frac{\partial\phi}{\partial u} + (y-vy-2xy)\frac{\partial\phi}{\partial v} = 0,$$

$$(1+xy)\frac{\partial\phi}{\partial y} + (2y+x^2-xu)\frac{\partial\phi}{\partial u} + (x-u-2y^2)\frac{\partial\phi}{\partial v} = 0.$$

The equations subsidiary to the integration of the first of these are

$$\frac{dx}{1+xy} = \frac{dy}{0} = \frac{du}{v+2x+xy} = \frac{dv}{y(1-v-2x)},$$

three independent integrals of which are

$$\begin{aligned} &y, \\ &\rho = v + y(u-x), \\ &\sigma = u - x(v+x); \end{aligned}$$

and therefore we have $\qquad \phi = f(y, \rho, \sigma).$

When this is substituted in the second equation, we have

$$\begin{aligned} &\frac{\partial f}{\partial y}(1+xy) \\ &\quad + \frac{\partial f}{\partial\rho}\left\{(1+xy)\frac{\partial\rho}{\partial y} + (2y+x^2-xu)\frac{\partial\rho}{\partial u} + (x-u-2y^2)\frac{\partial\rho}{\partial v}\right\} \\ &\quad + \frac{\partial f}{\partial\sigma}\left\{(1+xy)\frac{\partial\sigma}{\partial y} + (2y+x^2-xu)\frac{\partial\sigma}{\partial u} + (x-u-2y^2)\frac{\partial\sigma}{\partial v}\right\} = 0 \end{aligned}$$

or after reduction

$$\frac{\partial f}{\partial y}+2y\frac{\partial f}{\partial \sigma}=0.$$

The equations subsidiary to the integration of this equation are

$$\frac{dy}{1}=\frac{d\rho}{0}=\frac{d\sigma}{2y},$$

two independent integrals of which are

$$\rho,$$
$$\sigma-y^2;$$

and therefore $$f(y, \rho, \sigma)=\psi(\rho, \sigma-y^2),$$

where, for the most general solution, ψ is an arbitrary function.

Evidently all solutions are determinable by means of the two independent solutions ρ, $\sigma-y^2$; hence the integral equivalent of the original equations is

$$v+y(u-x)=a,$$
$$u-x(v+x)-y^2=\beta.$$

40. The result proved in § 39 as to the simultaneous equivalence of the systems (I) and (II) may be obtained as follows.

Any solution of the system (II) satisfies the equation

$$\mu_1\Delta_1\phi+\mu_2\Delta_2\phi+\ldots\ldots+\mu_m\Delta_m\phi=0,$$

where the coefficients μ are an arbitrary set of independent functions of the variables. Now a solution of this equation is a solution of the subsidiary set

$$\ldots\ldots=\frac{dx_t}{\mu_t}=\ldots\ldots=\frac{du_s}{\sum\limits_{t=1}^{m}\mu_t U_{st}}=\ldots\ldots;$$

and therefore every solution of the system (II) is a solution of these subsidiary equations, whatever be the values of the quantities μ. They become, after elimination of the μ's, the system

$$du_s=\sum_{t=1}^{m}U_{st}\,dx_t,$$

that is, the system (I); and therefore by the ordinary theory of partial differential equations* every solution of the system (II) is a solution of the system (I).

On account of this equivalence and of the fact that the partial

* *Treatise*, §§ 187, 189.

differential equations of the kind just considered, viz., homogeneous and linear in the differential coefficients, arise in some of the methods (e.g. in Clebsch's) of dealing with Pfaff's problem*, it is convenient to discuss in this connection some of their properties. The number of independent solutions of such a system has already been investigated in § 37; and one method of derivation of the solutions has been obtained, founded on that investigation. But because the systems of equations (I) and (II) are equivalent, it is natural to expect that Mayer's method for the ordinary equations can be extended to the integration of the partial equations; the extension is easily made as follows.

41. The system of partial differential equations being

$$\frac{\partial\phi}{\partial x_t} + \sum_{s=1}^{n} U_{st}\frac{\partial\phi}{\partial u_s} = 0, \quad (t = 1, 2, \ldots\ldots, m) \ldots\ldots\ldots\ldots(\text{II}),$$

we leave the variables u untransformed and transform the variables x by the substitutions of § 34, viz.

$$x_t = \alpha_t + (y_1 - \theta) f_t,$$

where $f_1, \ldots, f_m$ are m independent functions of the new m independent variables $y_1, \ldots, y_m$, and θ is a constant. Then we have

$$\begin{aligned}\frac{\partial\phi}{\partial y_1} &= \sum_{s=1}^{m}\frac{\partial\phi}{\partial x_s}\frac{\partial x_s}{\partial y_1}\\ &= \sum_{s=1}^{m}\left\{f_s + (y_1-\theta)\frac{\partial f_s}{\partial y_1}\right\}\frac{\partial\phi}{\partial x_s}\\ &= -\sum_{s=1}^{m}\left\{f_s + (y_1-\theta)\frac{\partial f_s}{\partial y_1}\right\}\sum_{r=1}^{n} U_{rs}\frac{\partial\phi}{\partial u_r}\\ &= -\sum_{r=1}^{n} Y_{r1}\frac{\partial\phi}{\partial u_r};\end{aligned}$$

and for values 2, 3, ..., m of q

* Another important set of such equations, similar in form to those here discussed and complete as a system, is the set of equations characteristic of the concomitants of quantics. See, for example, a memoir on "Systems of Ternariants that are algebraically complete," *Amer. Journ. of Math.*, vol. xii. (1889), pp. 1—60, 115—161; the ternariants are determined by such a set of equations, eight in number, and it is there shewn (§ 18) by considerations entirely different from those occurring in this chapter that there is a number of functionally independent solutions which is easily seen to agree with the number determined in § 38 of the text.

$$\frac{\partial \phi}{\partial y_q} = \sum_{s=1}^{m} \frac{\partial \phi}{\partial x_s} \frac{\partial f_s}{\partial y_q}$$

$$= (y_1 - \theta) \sum_{s=1}^{m} \frac{\partial f_s}{\partial y_q} \frac{\partial \phi}{\partial x_s}$$

$$= -(y_1 - \theta) \sum_{s=1}^{m} \frac{\partial f_s}{\partial y_q} \sum_{r=1}^{n} U_{rs} \frac{\partial \phi}{\partial u_r}$$

$$= - \sum_{r=1}^{n} Y_{rq} \frac{\partial \phi}{\partial u_r};$$

so that for all values 1, 2, ..., m of t we have

$$\nabla_t \phi = \frac{\partial \phi}{\partial y_t} + \sum_{r=1}^{n} Y_{rt} \frac{\partial \phi}{\partial u_r} = 0 \quad \text{..............(II)}'.$$

The systems (II) and (II)′ are equivalent to one another; and, as in § 34, the conditions of coexistence of the equations in (II)′ are satisfied, so that (II)′ is a complete system.

To integrate (II)′, so as to obtain the n independent solutions in its integral system, it is sufficient to obtain the n independent integrals of the equivalent system of n ordinary equations

$$du_r = \sum_{s=1}^{m} Y_{rs} dy_s.$$

It has already, in § 34, been seen that for this purpose it is sufficient to take the system of equations

$$du_r = Y_{r1} dy_1,$$

and to obtain n independent solutions of them in the form

$$\phi_p(u_1, \ldots\ldots, u_n, y_1, \ldots\ldots, y_m) = \text{constant},$$

the quantities $y_2, \ldots, y_m$ being supposed invariable. Then the system of independent integrals of the set of ordinary differential equations is

$$\phi_p(u_1, \ldots\ldots, u_n, y_1, y_2, \ldots\ldots, y_m) = \phi_p(c_1, \ldots\ldots, c_n, \theta, y_2, \ldots\ldots, y_m),$$

which are therefore n independent solutions of the system (II)′. The solution-system of (II) can be obtained by eliminating the variables y and replacing them in terms of the variables x; and if, in the above equations, any right-hand side take the form $\psi_p(c_1, \ldots, c_n, \alpha_1, \ldots, \alpha_m)$, then the corresponding integral of (II) is

$$\psi_p(u_1, \ldots\ldots, u_n, x_1, \ldots\ldots, x_m) = \psi_p(c_1, \ldots\ldots, c_n, \alpha_1, \ldots\ldots, \alpha_m).$$

Further, the equations

$$\frac{du_r}{Y_{r1}} = dy_1$$

(with $y_2, \ldots, y_m$ invariable), which lead to the construction of the functions ϕ_p, are the equations subsidiary to the derivation of the most general integral of the equation in (II)′ given by $t = 1$.

Hence we have the theorem :—

To obtain a set of n independent solutions of the complete system of equations

$$\frac{\partial\phi}{\partial x_t} + \sum_{s=1}^{n} U_{st} \frac{\partial\phi}{\partial u_s} = 0, \ (t = 1, 2, \ldots\ldots, m)$$

we transform the variables x by the substitutions

$$x_t = a_t + (y_1 - \theta) f_t (y_1, \ldots\ldots, y_m)$$

and construct the equation

$$\frac{\partial\phi}{\partial y_1} + \sum_{r=1}^{n} Y_{r1} \frac{\partial\phi}{\partial u_r} = 0.$$

The equations subsidiary to this, viz.,

$$dy_1 = \frac{du_1}{Y_{11}} = \ldots\ldots = \frac{du_n}{Y_{n1}}$$

(with $y_2, \ldots, y_m$ invariable), are to be integrated; they lead to n equations of the form

$$\phi_p (u_1, \ldots\ldots, u_n, y_1, \ldots\ldots, y_m) = \text{constant.}$$

Then the system of solutions required is obtained from the n equations

$$\phi_p (u_1, \ldots\ldots, u_n, y_1, y_2, \ldots\ldots, y_m) = \phi_p (c_1, \ldots\ldots, c_n, \theta, y_2, \ldots\ldots, y_m)$$

by replacing the variables y by their values in terms of the original variables x; and if any function $\phi_p (c_1, \ldots, c_n, \theta, y_2, \ldots, y_m)$ be a pure constant $\psi_p (c_1, \ldots, c_n, a_1, \ldots, a_m)$, then the corresponding solution is

$$\psi_p (u_1, \ldots\ldots, u_n, x_1, \ldots\ldots, x_m) = \psi_p (c_1, \ldots\ldots, c_n, a_1, \ldots\ldots, a_m).$$

As before, the forms of the functions f_t are at our disposal, subject to the limitations of independence. *The simplest substitutions appear to be*

$$x_1 = y_1,$$
$$x_t = a_t + (y_1 - a_1) y_t;$$

then the quantities Y_{r1} occurring in the subsidiary equations are

$$Y_{r1} = U_{r1} + \sum_{t=2}^{m} U_{rt} y_t.$$

Ex. To solve the system of equations

$$\left.\begin{array}{l} p_3 (x_4 - x_5) + p_1 (x_5 - x_6) + p_2 (x_6 - x_4) = 0 \\ p_4 (x_4 - x_5) + p_1 (x_5 - x_1) + p_2 (x_1 - x_4) = 0 \\ p_5 (x_4 - x_5) + p_1 (x_5 - x_2) + p_2 (x_2 - x_4) = 0 \\ p_6 (x_4 - x_5) + p_1 (x_5 - x_3) + p_2 (x_3 - x_4) = 0 \end{array}\right\},$$

which, as may easily be verified, are a complete system.

We take, to harmonise with the preceding notation,

$$\begin{array}{ll} x_1 = u_1, & x_4 = a_2 + (y_1 - a_1) y_2, \\ x_2 = u_2, & x_5 = a_3 + (y_1 - a_1) y_3, \\ x_3 = y_1, & x_6 = a_4 + (y_1 - a_1) y_4; \end{array}$$

then

$$\begin{array}{ll} \lambda U_{11} = (x_5 - x_6), & \lambda U_{21} = (x_6 - x_4), \\ \lambda U_{12} = (x_5 - x_1), & \lambda U_{22} = (x_1 - x_4), \\ \lambda U_{13} = (x_5 - x_2), & \lambda U_{23} = (x_2 - x_4), \\ \lambda U_{14} = (x_5 - x_3), & \lambda U_{24} = (x_3 - x_4), \end{array}$$

where λ is $x_4 - x_5$. Now

$$Y_{11} = U_{11} + y_2 U_{12} + y_3 U_{13} + y_4 U_{14},$$
$$Y_{21} = U_{21} + y_2 U_{22} + y_3 U_{23} + y_4 U_{24};$$

so that

$$Y_{11} + Y_{21} = -(1 + y_2 + y_3 + y_4),$$
$$x_4 Y_{11} + x_5 Y_{21} = -(x_6 + y_2 u_1 + y_3 u_2 + y_4 y_1).$$

The subsidiary equations being

$$dy_1 = \frac{du_1}{Y_{11}} = \frac{du_2}{Y_{21}}$$

with y_2, y_3, y_4 invariable, we have one integral given by

$$dy_1 = \frac{du_1 + du_2}{Y_{11} + Y_{21}} = -\frac{d(u_1 + u_2)}{1 + y_2 + y_3 + y_4},$$

and therefore

$$\begin{aligned} u_1 + u_2 + y_1 (1 + y_2 + y_3 + y_4) &= \text{constant} \\ &= c_1 + c_2 + a_1 (1 + y_2 + y_3 + y_4). \end{aligned}$$

From this we have

$$u_1 + u_2 + y_1 + y_2 (y_1 - a_1) + y_3 (y_1 - a_1) + y_4 (y_1 - a_1) = c_1 + c_2 + a_1$$

and therefore

$$u_1 + u_2 + x_3 + x_4 - a_2 + x_5 - a_3 + x_6 - a_4 = c_1 + c_2 + a_1.$$

Hence one integral of the system is

$$Z_1 = x_1 + x_2 + x_3 + x_4 + x_5 + x_6.$$

Again, we have

$$dy_1 = \frac{x_4 du_1 + x_5 du_2}{x_4 Y_{11} + x_5 Y_{21}} = -\frac{\{a_2 + (y_1 - a_1) y_2\} du_1 + \{a_3 + (y_1 - a_1) y_3\} du_2}{a_4 + y_2 u_1 + y_3 u_2 + 2y_4 y_1 - a_1 y_4},$$

the integral of which is easily seen to be

$$(a_4 - a_1 y_4) y_1 + y_1^2 y_4 + (y_2 u_1 + y_3 u_2)(y_1 - a_1) + a_2 u_1 + a_3 u_2 = \text{constant}$$
$$= a_4 a_1 + a_2 c_1 + a_3 c_2$$

on taking $y_1 = a_1$; hence an integral of the original system is

$$x_6 x_3 + x_5 u_2 + x_4 u_1 = a_4 a_1 + a_2 c_1 + a_3 c_2,$$

or is

$$Z_2 = x_1 x_4 + x_2 x_5 + x_3 x_6.$$

Now there are only two independent integrals of the original system, and two such have been obtained in Z_1 and Z_2. Hence the most general solution possible of the original set of simultaneous equations is

$$z = \phi(Z_1, Z_2),$$

where ϕ is any arbitrary function.

42. The class of equations just considered, homogeneous and linear in the partial differential coefficients and free from explicit occurrence of the dependent variable, arise in Clebsch's method in the reduction of a differential expression to its normal form, and in Jacobi's method for the integration of any partial differential equation of the first order. In each of these cases, what is wanted is not so much the complete system of solutions as a single solution.

Now when any quantity U_p, where

$$U_p = \phi_p(u_1, \ldots, u_n, y_1, \ldots, y_m) - \phi_p(c_1, \ldots, c_n, \theta, y_2, \ldots, y_m) = 0,$$

is substituted in the transformed differential equations, they may be satisfied either identically, or in virtue of $U_p = 0$, or in virtue of $U_p = 0$ combined with the other solutions in the integral-system. If then $U_p = 0$ be the only integral of the system which has been obtained, it cannot in the latter circumstances be regarded (when taken alone) as an integral of the differential equations; it does not in fact satisfy them, when it is the only integral known.

To meet this limited necessity of only a single integral, Mayer (l. c. § 5) gives a method of obtaining, by a process of differential derivation, at least one solution of the original system of partial

equations from any integral of the subsidiary system. Take any integral of the subsidiary system, say

$$\phi(u_1, \ldots\ldots, u_n, y_1, \ldots\ldots, y_m) = \text{constant};$$

then it has been proved that

$$U = \phi(u_1, \ldots, u_n, y_1, y_2, \ldots, y_m) - \phi(c_1, \ldots, c_n, \theta, y_2, \ldots, y_m) = 0$$

satisfies the system of equations (II)′ of § 41

$$\nabla_t U = 0 \ldots\ldots (t = 1, 2, \ldots\ldots, m).$$

Of these, $\nabla_1 U = 0$ is satisfied identically on account of the mode of derivation. The remainder may be satisfied identically; or in virtue of $U = 0$, in which case U is the required solution of the system of equations; or they may be satisfied only in virtue of other equations which, with $U = 0$, constitute an integral system.

In the last case let a new equivalent equation

$$c_1 = U_1(u_1, \ldots\ldots, u_n, y_1, \ldots\ldots, y_m, \theta, c_2, \ldots\ldots, c_n)$$

be derived from $U = 0$ by solving algebraically for c_1. Then we still have

$$\nabla_1 U_1 = 0$$

satisfied identically; and the other equations are, for $t = 2, \ldots, m$,

$$\nabla_t U_1 = 0,$$

which are to be satisfied in virtue of the (at present) unknown equations in the integral system. Since all the equations $\nabla_t U_1 = 0$ are thus not identical and since they do not involve c_1, so that they cannot be satisfied in virtue of $c_1 = U_1$, we shall be able to deduce from them the values of some of the constants c, say of $c_2, \ldots, c_h$, in terms of u and y and the remaining constants c; let them be

$$c_i = U_i(u_1, \ldots\ldots, u_n, y_1, \ldots\ldots, y_m, \theta, c_{h+1}, \ldots\ldots, c_n)$$

for $i = 2, \ldots, h$. Since these equations are the equivalents of some of the integral equations, they satisfy the set (II)′ of differential equations. These are treated in the same manner as the equation $c_1 = U_1$; and so we proceed either until we obtain a common solution of the equations or until we can express, from the non-identically satisfied equations, the values of all the quantities c in terms of y, u, θ alone. These, when taken in the aggregate, constitute an integral system.

But, further, when any one of them, say

$$c_r = U_r(u_1, \ldots\ldots, u_n, y_1, \ldots\ldots, y_m, \theta),$$

is substituted in the equations (II)′, then all the equations $\nabla_t U_r = 0$ are satisfied identically; for none of the constants c enter into an equation $\nabla_t U_r = 0$, so that no one of the n equations (each of which involves one of those constants) can be useful in making $\nabla_t U_r$ vanish. Hence each of the members of the system $U_1, \ldots, U_n$ in the obtained form is, by itself, a solution of all the differential equations.

It thus follows that *from any integral of the subsidiary system at least one solution of the partial equations can be derived:* and its explicit form, as a solution of the original system (II) of equations, is immediately given by a transformation to the original variables through the equations of substitution.

Moreover when the value θ is assigned to y_1 in the function U_r, it becomes simply u_r, because the values of the variables $u_1, \ldots, u_n$ for this value of y_1 are to be given by the equations as $c_1, \ldots, c_n$ respectively. The integrals, in the form just indicated in the proof of Mayer's theorem, are a system called principal integrals*; and the values of $x_1, \ldots, x_m$ for $y_1 = \theta$ are $a_1, \ldots, a_m$. Hence it follows that there exists for the system (II) of differential equations a set $U_1, \ldots, U_n$ of principal integrals, such that for properly chosen constant values $a_1, \ldots, a_m$ of $x_1, \ldots, x_m$ they assume the values $u_1, \ldots, u_n$†.

43. When Clebsch's first method (Chap. VIII.) is applied to the reduction of an unconditioned linear differential expression in $2n$ variables, the determination of each successive element of the normal form—or, what is the same thing, of each integral of the Pfaffian equation—enables the number of variables in the original expression to be diminished by unity. Thus when μ integrals have been obtained, there are $2n-\mu$ variables left: and since $n-\mu$ integrals remain to be found (that is, the expression modified by the integrals so as to contain $2n-\mu$ variables has a normal form containing only $n-\mu$

* Natani, *Crelle*, t. lviii. p. 302.

† See Lie, "Theorie des Pfaff'schen Problems," *Arch. f. Math. og Nat.*, t. ii. (1877), pp. 338—379; § 34.

It may be remarked that the only limitation on the choice of the constant values of x is that they shall not lead to indeterminate or infinite values of functions which occur.

differential elements) it follows (§§ 122—124) that the $(\mu+1)^{\text{th}}$ integral satisfies

$$1+(2n-\mu)-2(n-\mu)$$

$(=\mu+1)$ differential equations of the kind just considered; and these equations involve the $2n-\mu$ variables. Hence (§ 38) the number of integrals which they possess and which are independent of one another is

$$2n-\mu-(\mu+1),$$

that is, is $2n-2\mu-1$. Now what is needed on the adoption of Mayer's method of integration is a single solution of the associated subsidiary system of ordinary linear equations which are $2n-2\mu-1$ in number; so that, for the determination of the $(\mu+1)^{\text{th}}$ integral of the Pfaffian equation, Mayer's method requires only one solution of a system of $2n-2\mu-1$ ordinary linear equations of the first order. When we take $\mu=0, 1, \ldots, n-1$, being the values which give all the integrals, it follows that by Mayer's method the Pfaffian problem is completely solved by the determination of a single integral of each one of a system of

$$2n-1,\ 2n-3,\ \ldots\ldots,\ 3,\ 1$$

ordinary differential equations of the first order.

Similar considerations, to determine the number of integrations necessary and sufficient, apply when Jacobi's method of integration of a partial differential equation of the first order is used.

CHAPTER III.

Historical Summary of Methods of treating Pfaff's Problem.

44. The total equations hitherto considered are such as may be derived from a single integral equation; and their coefficients satisfy a certain number of conditions, which are both necessary and sufficient to ensure the possibility of that derivation. But an equation among the small variations of the variables may subsist though only some, or even none, of the conditions of derivation from a single equation are satisfied; and a question arises as to the form of the integral equivalent of such an equation, if there be an integral equivalent.

45. Euler, who regarded each differential equation as necessarily derived from an integral equation, declared* that, unless the conditions be satisfied, the equation is absurd and has no significance. Monge† however pointed out that the absurdity lies, not in the supposition that the equation can have significance, but in the inference that the integral equivalent consists of a single equation. And he illustrated his statement by the remark that the total differential equation in three variables, if the condition be satisfied, belongs to a surface, but that, if the condition be not satisfied, it represents some property of a tortuous curve‡ though the curve itself requires two integral equations for its full expres-

* *Inst. Calc. Int.*, Vol. iii., Part 1, § 1, c. 1 (2nd edition), p. 5.

† *Mém. de l'Acad. Royale des Sciences* (1784), p. 535.

‡ The property represented by the equation is, when the condition is not satisfied, one common to a family of tortuous curves defined by two integral equations; but, when the condition is satisfied, the family of tortuous curves is constituted by all the curves which can be drawn on some surface, and the differential equation which represents the property can be regarded as belonging to the surface.

sion; and the differential equation is satisfied by means of the two integral equations. And Monge inferred that an integral equivalent of a total differential equation in any number of variables can be constituted by a system of equations, the number in the system being never greater, and sometimes less, than the number of variables diminished by unity.

46. In this condition the theory of these equations remained until 1815 when Pfaff presented to the Academy of Berlin his classical memoir* in which he gave the result that an integral equivalent of a total differential equation, containing $2n$ or $2n-1$ variables, can always be constituted by a system of integral equations, the number in the system being not greater than n. It is on account of the importance of the results first announced in this memoir that the problem of determining the integral equivalent of an unconditioned total differential equation is associated with the name of Pfaff.

So far as concerns the equation in an odd number of variables Pfaff merely stated the above result relative to the number of equations in the integral system, which was apparently inferred as a generalisation from a few individual instances; Gauss† merely repeated Pfaff's statement; the lacuna was first supplied by Jacobi‡ who gave a proof of the statement. Some improvements and amplifications are due to Gauss and Jacobi§; and the details of the process have been rendered much easier by investigations of Cayley relative to skew determinants‖; in them certain functions occur which had already occurred in Pfaff's investigations and the functions are therefore called Pfaffians.

The method introduced by Pfaff for the construction of the integral system depends upon the gradual reduction of the number of differential elements in the equation; and each reduction of this number by one unit is effected by means of the solution of systems of ordinary simultaneous equations.

* "Methodus generalis aequationes differentiarum partialium nec non aequationes differentiales vulgares, utrasque primi ordinis, inter quotcunque variabiles complete integrandi." *Abh. d. K.-P. Akad. d. Wiss. zu Berlin* (1814-5), pp. 76—136.

† *Gött. gel. Anz.* (1815), pp. 1025—1038; *Ges. Werke*, t. iii., pp. 231—241.

‡ *Crelle*, t. ii. (1827), pp. 347—357; *Ges. Werke*, t. iv., pp. 17—29.

§ In this connection (and also in connection with Chapter IV.) a memoir by Mayer, *Math. Ann.*, t. xvii. (1880), pp. 523—530 may be consulted.

‖ For references, see Scott's *Theory of Determinants*.

There are thus a number of systems of subsidiary equations to be integrated when Pfaff's original method is used. Among the improvements due to Jacobi already referred to, the most important and essential improvement is connected with these integrations. He shewed* that the introduction of "initial values" of the variables—afterwards used by Lie (Chap. X.) with great effect in the theory of the Pfaffian problem—renders it possible to take the integrals of the first subsidiary system in a form, which leads immediately to the transformation of the equation: but this simplification was effected by him only for the unconditioned equation in an even number of variables. And in the particular case when the Pfaffian expression occurs in connection with the integration of a partial differential equation of the first order, he shewed that the integration of the first subsidiary system is sufficient for the reduction of the Pfaffian expression to its normal form and is therefore sufficient for the integration of the original partial differential equation.

47. No substantial additions† to the development of the theory were made until the publication in 1861 and 1862 of the memoirs of Natani‡ and Clebsch§ and of what is practically the second edition of Grassmann's Ausdehnungslehre||. It may be inferred

* *Crelle*, t. xvii. (1837), pp. 97—162; *Ges. Werke*, t. iv., pp. 57—127.

† Mention should however be made of a memoir by Frisiani, "*Sull' integrazione delle equazioni differenziali ordinarie di primo ordine e lineari fra un numero qualunque di variabili*," published in 1847 as an appendix to the *Effemeridi Astronomiche di Milano* for the year 1848. In that memoir he gives an account of the theory as it was known at that date, unfortunately without a single reference to other writers. He indicates the Gaussian transformation (§ 68 post) to the reduced form of a Pfaffian expression: he solves the subsidiary equations obtained in the form, given in § 55 (post), so as to have them in the form simplest for integration; he discusses the possibility of having the Pfaffian equation satisfied by equations fewer than the canonical number, when relations among the coefficients of the differential elements exist; and he applies the theory to the integration of the partial differential equation of the first order.

‡ *Crelle*, t. lviii., pp. 301—328, dated January, 1860.

§ *Crelle*, t. lx., pp. 193—251, t. lxi., pp. 146—179, dated September, 1860.

|| "Die Ausdehnungslehre, vollständing und in strenger Form bearbeitet" von Hermann Grassmann; Berlin, 1862.

The date of the completion of the sections relating to the present subject can only be inferred as previous to August 1861, the date of the preface, and as subsequent to 1844, because he implies in that preface that the additions (which include these sections) were the work of the intervening seventeen years. His

from a statement of Jacobi's* that he was in 1845 in possession of some such additions; but as his statement is hereafter (§ 68) shewn to be a mere deduction from Pfaff's general result, no sure inference in this respect can be framed. Certainly, subsequently to this isolated passage in a memoir dealing with another subject, he published nothing which bears directly upon the theory of Pfaff's problem; and, after his death, nothing was found among his manuscript papers relating to it.

48. Grassmann's method is more difficult of comprehension† on account of the unfamiliar character of the analysis used; the following are the main features of his theory.

He first expresses the differential equation in m variables in a form $Xdx = 0$ where the extensive variable x involves m units. He proves that, if the equivalent integral system contain n equations of the form $u = c$, then

$$Xdx = \Sigma U du;$$

and he expresses the conditions, necessary and sufficient, for the existence of these n integrals—conditions which lead to the inference that $2n - 1$ must be less than m‡.

In the case when m is equal to $2n$, so that the equation is unconditioned, the first step is, as usual, the construction of a

investigations on the Pfaffian equation were probably among the latest finished, because he requires some of the later developments of his analysis for the consideration of that equation—developments, indeed, which from their position (§§ 504—510) appear to have been specially made for this purpose.

In the historical summary in the text I have given first an abstract of Grassmann's results—not because it is clear that he completed his investigations earlier than Natani or than Clebsch, but because they naturally come in this position in the gradual development of the theory. It is perhaps superfluous to remark that the substance and the form of the results of Grassmann, Natani and Clebsch, taken in conjunction with the dates of publication, are sufficient to prove the complete independence of their investigations.

* *Crelle*, t. xxix., p. 253.

† See note at beginning of Chapter V.

‡ It is assumed implicitly that, if the coefficients of an equation satisfy no characteristic condition, then the number of variables is even; so that Grassmann practically considers only the even classes of unconditioned equations. Of course, from the point of view of the general theory of the Pfaffian equation, this is a defect; but it is explicable by the fact that with him, as with Pfaff, the equation has its origin in the partial differential equation of the first order, in which case the number of variables is even. (See Chapter VII.)

subsidiary equation the integral of which makes the transition to a new form $Ada = 0$ possible; in this form a is a new extensive variable involving only $2n - 1$ units. Then by assuming one integral, exactly in the same manner as earlier writers above referred to had already done, he passes to an unconditioned equation in $2n - 2$ variables. The former process is now repeated; and so the n integrals are gradually obtained.

In the case when m is greater than $2n$, he shews that the set of m subsidiary equations—the "numerical" equivalent of his single extensive equation—contains only $2n$ independent equations. By the justifiable assumption that $m - 2n$ of the numerical variables are constant, he obtains a transforming relation which, applied to $Xdx = 0$, changes it into an equation $Ada = 0$, where the extensive variable contains only $m - 1$ units and for which, if $m - 1 > 2n$, the conditions that the integral system is composed of n equations are satisfied. This process is again applied and continued until finally he arrives at an unconditioned equation, the extensive variable of which involves only $2n$ units. To this equation the earlier method applies; and the system of n integrals is thus gradually obtained.

The limitation of the method to equations in an even number of variables, if they be unconditioned, has already been referred to in the preceding note; and a practical weakness—the integration of the subsidiary system—will be referred to hereafter (p. 137). One distinct advance contributed by the method is the indication of a process for an equation, the coëfficients of which satisfy conditions reducing the number of integrals below that required by the most general equation in the same number of variables; another is the expression of the conditions which are necessary and sufficient for this purpose. He does not, however, prove that his conditions are independent of one another; and he does not see that the satisfaction of some accidental conditions, viz., those of the vanishing of the interrupted product $\left[\left(\frac{dX}{dx}\right)^n\right]$, would modify the reduced form. This last possibility was only effected by later writers, Clebsch and Lie for instance, who discussed unconditioned equations in an odd number of variables.

It is, further, only proper that attention should be drawn to the remarkable formal conciseness of his results.

49. Natani and Clebsch adopt, for the solution of the problem, methods which, though different in detail, have their characteristic idea common; and on this basis a comparison of the methods has been made by Hamburger*. The fundamental idea of each method is the gradual reduction of the number of differential elements in the equation, not as in Pfaff's method by successive transformations, but by means of the successive members of the integral system equivalent to the differential equation; and the number of subsidiary equations, which have to be solved, is considerably less than in Pfaff's process. Thus a differential equation in $2n$ variables, having n equations in its equivalent integral system, is reduced by means of one of those integrals (considered as simultaneous with it) to a differential equation in $2n-1$ variables, having only $n-1$ equations in its equivalent integral system. The new differential equation is similarly reduced, by means of one of the integrals in its equivalent system, to a differential equation in $2n-2$ variables, having only $n-2$ equations in its integral system. And so on in order, until a differential equation in $n+1$ variables is obtained, having a single integral as its equivalent; it is therefore of the kind already considered (Chap. I.).

Of the methods due to these two contributors to the theory, that which has been proposed by Natani is the simpler, and it is the more direct for the construction of a particular integral system; it has moreover the advantage of being unhampered by the accidental difficulties which arise when superfluous conditions are satisfied. But it is characterised by a not quite complete generalisation. This deficiency has been partly supplied by the methods of Clebsch, who shews how to obtain, from any special integral system—such as Natani's for example—the most general integral system. Clebsch has given two methods. His first method is similar in scope to Natani's but it is not so effective for equations in an odd number of variables, (in fact, he nowhere gives an adequate discussion of this class) and the process remains one of gradual reduction. His second method, which is comparatively simple in its results and is powerful, is unfortunately limited in his investigations to unconditioned equations in an even number

* *Grunert's Archiv*, t. lx. (1877), pp. 185—214.

of variables. In both of Clebsch's methods the results are founded, after long and laborious analysis, upon systems of simultaneous partial differential equations.

Thus the salient feature of Natani's method is the effective determination of some system of integral equations corresponding to the differential equation; the salient feature of Clebsch's method is the *a posteriori* generalisation of such a system and, for an unconditioned equation in an even number of variables, the *a priori* determination of such a general system.

Both Clebsch and Natani in their respective ways have considered the effect of the knowledge of one or more integrals of the differential equation upon the form of the remaining integrals; in this regard Clebsch's results are the more explicit, as his method is better suited to its determination.

Among the chief desiderata of this part of the theory are extensions of Clebsch's second (and general) method, first to conditioned equations in an even number of variables and second to equations, whether conditioned or not, in an odd number of variables.

50. These desiderata were indirectly and partially supplied by Lie, who worked from a different stand-point*. He made the theory of tangential transformations† the basis of his investigations, especially in regard to the transformation of the differential expression. He established the persistence of character of the normal form, an invariantive property which had been assumed by Clebsch; and the equations which give the relation of two equivalent normal forms are obtained and agree with those which Clebsch gave. The criteria which determine the number of functions in, and therefore the character of, the normal form are found; and, when these are once known for any expression, then Lie's method reduces the expression to an equivalent unconditioned expression with a similar normal form. This reduction is made by a number of substitutions of the type originally used by Cauchy (1819), and afterwards by Hamilton, Jacobi, and Mayer.

* His papers were published at various times in 1873 and 1874; the most convenient summary of the results is his memoir "Theorie des Pfaff'schen Problems," *Arch. for Math. og Nat.*, t. ii. (1877), pp. 338—379.

† Of this theory Mayer gave an independent establishment; see Chapter IX.

The first element of the normal form of the new unconditioned expression is determined by a partial differential equation; and this element is then used to modify the expression into one in fewer variables. The normal form is gradually built up by a series of alternate determinations of a new element and reductions of expressions, made conditioned by the use of this element, into unconditioned expressions. When the normal form of the unconditioned expression, equivalent to the original, has been obtained, the transition to the normal form of the latter is a matter of definite re-substitution; and the integral system of the given equation is then inferred from a theorem indicated first, I think, by Grassmann in its usual form.

Lie's results constitute a distinct addition to the theory. The whole of his investigation is not, it could hardly be expected to be, novel; but in his exposition there lies a great interest in the application and combination of ideas which occur in other associations.

51. About the time of publication of the memoir by Lie just referred to, Frobenius had [Sept. 1876] completed his memoir* dealing with Pfaff's problem. He discusses the theory of the normal form rather than the integration of the equation; and the analysis is more algebraical than differential. He obtains the bilinear covariant associated with the Pfaffian expression: and then changes the Pfaffian and the differential covariant into homogeneous algebraical forms subject to linear transformations. The persistence of a certain invariantive integer associated with a pair of characteristic determinants is shewn to be necessary and sufficient for the transformation of one Pfaffian into another; the equations of substitution being connected with the normal form. He thus indirectly arrives at the normal form; its character is uniquely determined by the value of the invariantive integer. A concise statement of the principal results is given in § 168.

The novel interest of the method lies chiefly in the connection of the number of the terms in the normal form with the critical algebraical conditions, which lead to Clebsch's differential equations.

* "Ueber das Pfaff'sche Problem," *Crelle*, t. lxxxii. (1877), pp. 230—315.

52. The work which M. Darboux was carrying on at this time [1877] has relations with both that of Lie and that of Frobenius. His method of simultaneous sets of variations of the independent variables is substantially the same as that of Frobenius who considers two sets of cogredient variables; and these simultaneous variations are used to establish part of Lie's theory of tangential transformations. But all his results relate to the theory of equivalent forms of the equation and not to its integral system; his memoir* was not published until 1882 and then all his results had been anticipated; and therefore I have given merely a short statement of those of his propositions which deal with the equation.

* "Sur le problème de Pfaff," *Darb. Bull.*, 2^me Sér. t. vi. (1882), pp. 14—36 49—68.

CHAPTER IV.

Pfaff's method of reduction, completed as by Gauss and Jacobi.

53. The most general form of the total differential equation of the first order and the first degree in p variables is

$$\Omega = X_1 dx_1 + X_2 dx_2 + \ldots + X_p dx_p = 0 \text{..........(1)},$$

where $X_1, X_2, \ldots, X_p$ are functions of the variables $x_1, x_2, \ldots, x_p$; and, as the class of equations $\Omega = 0$, which can be satisfied by means of a single integral equation has already been discussed, it will be assumed that the equations of relation among the quantities X, which imply the derivation of the differential equation from a single integral equation, are not all satisfied. If there be an integral equivalent, that is, some set of relations free from differential elements, in virtue of which the equation $\Omega = 0$ can be satisfied, the first step in the investigation of that equivalent will naturally be the reduction of the differential equation to the explicitly simplest form which it can assume. We proceed to prove that *it can be transformed in all cases so as to involve not more than $\frac{1}{2}p$ or $\frac{1}{2}(p+1)$ differential elements, according as p is even or odd.*

54. Let $p-1$ functions $u_1, u_2, \ldots, u_{p-1}$ of the variables be introduced, with at present only the single assumption that there is no functional relation among them; it follows from this independence that all but one of the original variables, say x_p, can be expressed in terms of that one and of $u_1, u_2, \ldots, u_{p-1}$, by equations

$$x_r = x_r(u_1, u_2, \ldots, u_{p-1}, x_p) \text{.................(2)}$$

for $r = 1, 2, \ldots, p-1$. Now let Ω be transformed by means of the relations (2); it takes the form

$$\Omega = Y_1 du_1 + Y_2 du_2 + \ldots + Y_{p-1} du_{p-1} + Y_p dx_p,$$

$$\left.\begin{array}{l} \text{where} \qquad Y_r = X_1 \dfrac{\partial x_1}{\partial u_r} + X_2 \dfrac{\partial x_2}{\partial u_r} + \ldots + X_{p-1} \dfrac{\partial x_{p-1}}{\partial u_r} \\ \text{for } r = 1, 2, \ldots, p-1, \text{ and} \\ \qquad Y_p = X_1 \dfrac{\partial x_1}{\partial x_p} + X_2 \dfrac{\partial x_2}{\partial x_p} + \ldots + X_{p-1} \dfrac{\partial x_{p-1}}{\partial x_p} + X_p \end{array}\right\} \ldots\ldots(3).$$

The forms of the functions u are at our disposal and can, in general, be used to satisfy $p-1$ independent conditions, provided the conditions are not inconsistent with one another; and, on account of the assumption made as to the forms of the functions u, the same principle applies to the $p-1$ functions in (2).

As a first condition then let the functions be so chosen that the coefficient of dx_p in the transformed value of Ω vanishes; this requires

$$X_1 \frac{\partial x_1}{\partial x_p} + X_2 \frac{\partial x_2}{\partial x_p} + \ldots + X_{p-1} \frac{\partial x_{p-1}}{\partial x_p} + X_p = 0 \ldots\ldots\ldots(4).$$

As the remaining $p-2$ conditions, let the forms of the functions x_r in (2) be so chosen that the ratios $Y_r : Y_1$ for the values $2, 3, \ldots, p-1$ of r (being $p-2$ in number) are independent of x_p. If this be the case, the quantity x_p can occur in $Y_1, Y_2, \ldots, Y_{p-1}$ only by occurrence in a factor M common to all the quantities Y, so that we may have

$$Y_r = M U_r \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

where U_r involves only $u_1, u_2, \ldots, u_{p-1}$ at the utmost and does not involve x_p, while x_p if it occur at all in Y_r must do so only in the factor M. Since U_r is independent of x_p, we have

$$\frac{\partial Y_r}{\partial x_p} = \frac{\partial M}{\partial x_p} U_r;$$

and therefore the quantities Y_r which occur in (3) are such that

$$\frac{1}{Y_r} \frac{\partial Y_r}{\partial x_p} = \frac{1}{M} \frac{\partial M}{\partial x_p} = \mu,$$

or say

$$\frac{\partial Y_r}{\partial x_p} = \mu Y_r \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

for the values 1, 2, 3, ..., $p-1$. And then the system of p equations (4) and (6) is to be satisfied by means of the $p-1$ functions, which occur in (2), and the quantity μ. Unless the system of equations be either inconsistent or subject to identical relations, they are sufficient for the determination of the functions and the quantity μ, which, when determined, transform the differential equation to

$$\Omega = M\left(U_1 du_1 + U_2 du_2 + \ldots + U_{p-1} du_{p-1}\right) = 0,$$

where $U_1, U_2, \ldots, U_{p-1}$ are functions of $u_1, u_2, \ldots, u_{p-1}$ alone. We proceed therefore to the consideration of the equations (4) and (6).

55. In Y_r, which by (3) is given as

$$X_1 \frac{\partial x_1}{\partial u_r} + X_2 \frac{\partial x_2}{\partial u_r} + \ldots + X_{p-1} \frac{\partial x_{p-1}}{\partial u_r},$$

the original variables $x_1, x_2, \ldots, x_{p-1}$ wherever they occur in the quantities X are to be replaced by their values as functions of $u_1, u_2, \ldots, u_{p-1}, x_p$; so that Y_r (which, in its first form, is an explicit function of $x_1, x_2, \ldots, x_{p-1}, x_p$ and the $p-1$ derivatives with respect to u_r) is now to be regarded as involving those $p-1$ derivatives and x_p explicitly, and the quantities $u_1, u_2, \ldots, u_{p-1}, x_p$ implicitly through their introduction instead of $x_1, x_2, \ldots, x_{p-1}$.

Taking now any one of the quantities Y we may write

$$Y = X_1 \frac{\partial x_1}{\partial u} + X_2 \frac{\partial x_2}{\partial u} + \ldots + X_{p-1} \frac{\partial x_{p-1}}{\partial u},$$

where the (unexpressed) subscript index of Y is the same as that of u. Taking the derivative of both sides with regard to x_p and remembering that, in consequence of the above explanation, x_p occurs in X explicitly (on account of its original occurrence) and implicitly (on account of its introduction through the substitutions for $x_1, x_2, \ldots, x_{p-1}$), we have

$$\begin{aligned}\frac{\partial Y}{\partial x_p} = {} & X_1 \frac{\partial^2 x_1}{\partial u \partial x_p} + X_2 \frac{\partial^2 x_2}{\partial u \partial x_p} + \ldots + X_{p-1} \frac{\partial^2 x_{p-1}}{\partial u \partial x_p} \\ & + \sum_{s=1}^{p-1} \frac{\partial x_s}{\partial u} \left(\frac{\partial X_s}{\partial x_1} \frac{\partial x_1}{\partial x_p} + \frac{\partial X_s}{\partial x_2} \frac{\partial x_2}{\partial x_p} + \ldots + \frac{\partial X_s}{\partial x_{p-1}} \frac{\partial x_{p-1}}{\partial x_p} + \frac{\partial X_s}{\partial x_p} \right).\end{aligned}$$

But by equation (4) we have

$$0 = X_p + X_1 \frac{\partial x_1}{\partial x_p} + X_2 \frac{\partial x_2}{\partial x_p} + \ldots + X_{p-1} \frac{\partial x_{p-1}}{\partial x_p};$$

and therefore, again remembering that u occurs implicitly in X on account of its introduction through the substitutions for $x_1, x_2, \ldots, x_{p-1}$,

$$0=\frac{\partial X_p}{\partial x_1}\frac{\partial x_1}{\partial u}+\frac{\partial X_p}{\partial x_2}\frac{\partial x_2}{\partial u}+\ldots+\frac{\partial X_p}{\partial x_{p-1}}\frac{\partial x_{p-1}}{\partial u}$$

$$+\sum_{t=1}^{p-1}\frac{\partial x_t}{\partial x_p}\left(\frac{\partial X_t}{\partial x_1}\frac{\partial x_1}{\partial u}+\frac{\partial X_t}{\partial x_2}\frac{\partial x_2}{\partial u}+\ldots+\frac{\partial X_t}{\partial x_{p-1}}\frac{\partial x_{p-1}}{\partial u}\right)$$

$$+X_1\frac{\partial^2 x_1}{\partial u\partial x_p}+X_2\frac{\partial^2 x_2}{\partial u\partial x_p}+\ldots+X_{p-1}\frac{\partial^2 x_{p-1}}{\partial u\partial x_p}.$$

Substituting from this equation for the last group of terms, which group occurs in $\frac{\partial Y}{\partial x_p}$, we have

$$\frac{\partial Y}{\partial x_p}=\sum_{s=1}^{p-1}\sum_{t=1}^{p-1}\frac{\partial x_s}{\partial u}\frac{\partial x_t}{\partial x_p}\frac{\partial X_s}{\partial x_t}+\sum_{s=1}^{p-1}\frac{\partial X_s}{\partial x_p}\frac{\partial x_s}{\partial u}$$

$$-\sum_{s=1}^{p-1}\frac{\partial X_p}{\partial x_s}\frac{\partial x_s}{\partial u}-\sum_{t=1}^{p-1}\sum_{s=1}^{p-1}\frac{\partial x_t}{\partial x_p}\frac{\partial x_s}{\partial u}\frac{\partial X_t}{\partial x_s}.$$

When the first and last summations are combined then

$$\frac{\partial Y}{\partial x_p}=\sum_{s=1}^{p-1}\sum_{t=1}^{p-1}a_{s,t}\frac{\partial x_s}{\partial u}\frac{\partial x_t}{\partial x_p}+\sum_{s=1}^{p-1}a_{s,p}\frac{\partial x_s}{\partial u},$$

where $a_{s,t}$ is defined as before by the equation

$$a_{s,t}=\frac{\partial X_s}{\partial x_t}-\frac{\partial X_t}{\partial x_s}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\quad(7),$$

or finally

$$\frac{dY}{\partial x_p}=\sum_{s=1}^{p-1}\frac{\partial x_s}{\partial u}\left\{a_{s,p}+\sum_{t=1}^{p-1}\left(a_{s,t}\frac{\partial x_t}{\partial x_p}\right)\right\}.$$

But by (6)
$$\frac{\partial Y}{\partial x_p}=\mu Y$$
$$=\mu\sum_{s=1}^{p-1}X_s\frac{\partial x_s}{\partial u},$$

so that
$$\mu\sum_{s=1}^{p-1}X_s\frac{\partial x_s}{\partial u}=\sum_{s=1}^{p-1}\left\{a_{s,p}+\sum_{t=1}^{p-1}\left(a_{s,t}\frac{\partial x_t}{\partial x_p}\right)\right\}\frac{\partial x_s}{\partial u}.$$

This equation holds when $u=u_1, u_2, \ldots, u_{p-1}$; so that it represents a system of equations which are linearly homogeneous in certain quantities of the type

$$\Theta_s=a_{s,p}-\mu X_s+\sum_{t=1}^{p-1}\left(a_{s,t}\frac{\partial x_t}{\partial x_p}\right),$$

the equations being

$$\Theta_1 \frac{\partial x_1}{\partial u_r} + \Theta_2 \frac{\partial x_2}{\partial u_r} + \ldots\ldots + \Theta_{p-1} \frac{\partial x_{p-1}}{\partial u_r} = 0$$

for $r = 1, 2, \ldots, p-1$. Now the determinant of the coefficients of the quantities Θ is not zero; for, by hypothesis, the functions in (2) are independent of one another and therefore

$$\frac{\partial (x_1, x_2, \ldots\ldots, x_{p-1})}{\partial (u_1, u_2, \ldots\ldots, u_{p-1})}$$

does not vanish. The system of equations therefore can only be satisfied by making each of the quantities Θ zero; and therefore we have

$$\left.\begin{array}{l}
\mu X_1 = a_{1,p} \phantom{+ a_{21}\frac{\partial x_1}{\partial x_p}} + a_{12}\frac{\partial x_2}{\partial x_p} + a_{13}\frac{\partial x_3}{\partial x_p} + \ldots + a_{1,p-1}\frac{\partial x_{p-1}}{\partial x_p} \\
\mu X_2 = a_{2,p} + a_{21}\frac{\partial x_1}{\partial x_p} \phantom{+ a_{12}\frac{\partial x_2}{\partial x_p}} + a_{23}\frac{\partial x_3}{\partial x_p} + \ldots + a_{2,p-1}\frac{\partial x_{p-1}}{\partial x_p} \\
\mu X_3 = a_{3,p} + a_{31}\frac{\partial x_1}{\partial x_p} + a_{32}\frac{\partial x_2}{\partial x_p} \phantom{+ a_{13}\frac{\partial x_3}{\partial x_p}} + \ldots + a_{3,p-1}\frac{\partial x_{p-1}}{\partial x_p} \\
\cdot\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\mu X_{p-1} = a_{p-1,p} + a_{p-1,1}\frac{\partial x_1}{\partial x_p} + a_{p-1,2}\frac{\partial x_2}{\partial x_p} + a_{p-1,3}\frac{\partial x_3}{\partial x_p} + \ldots \\
\text{and} \\
\mu X_p = \phantom{a_{p,p}+} a_{p,1}\frac{\partial x_1}{\partial x_p} + a_{p,2}\frac{\partial x_2}{\partial x_p} + a_{p,3}\frac{\partial x_3}{\partial x_p} + \ldots + a_{p,p-1}\frac{\partial x_{p-1}}{\partial x_p}
\end{array}\right\}\ldots(8),$$

the last equation being obtained from (4) and the first $p-1$ equations in (8), because

$$\begin{aligned}
\mu X_p &= -\mu X_1 \frac{\partial x_1}{\partial x_p} - \mu X_2 \frac{\partial x_2}{\partial x_p} - \ldots\ldots - \mu X_{p-1} \frac{\partial x_{p-1}}{\partial x_p} \\
&= -a_{1,p} \frac{\partial x_1}{\partial x_p} - a_{2,p} \frac{\partial x_2}{\partial x_p} - \ldots\ldots - a_{p-1,p} \frac{\partial x_{p-1}}{\partial x_p},
\end{aligned}$$

since all the terms of the second degree in the first derivative of $x_1, x, \ldots, x_{p-1}$ with respect to x_p disappear; ~~and~~, because $a_{s,t} = -a_{t,s}$, this equation is the same as the last of (8). The whole system of equations may be made more regular in form; for by substituting

$$\frac{1}{\mu} = y_p, \qquad \frac{1}{\mu} \frac{\partial x_r}{\partial x_p} = y_r \ldots\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

they become

$$\left.\begin{array}{l} \qquad\quad a_{12}y_2+a_{13}y_3+\ldots+a_{1,p-1}y_{p-1}+a_{1,p}y_p=X_1\\ a_{21}y_1+\qquad\quad +a_{23}y_3+\ldots+a_{2,p-1}y_{p-1}+a_{2,p}y_p=X_2\\ a_{31}y_1+\ a_{32}y_2\qquad\quad +\ldots+a_{3,p-1}y_{p-1}+a_{3,p}y_p=X_3\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ a_{p,1}y_1+a_{p,2}y_2+a_{p,3}y_3+\ldots+a_{p,p-1}y_{p-1}\qquad\quad =X_p \end{array}\right\}\ldots(10).$$

56. This system of equations is a system of p algebraical equations involving p quantities in the first degree; and, unless the equations are inconsistent or are subject to identical relations, they will determine these quantities $y_1, y_2, \ldots, y_p$. Let the determinant of the coefficients on the left-hand side be denoted by Δ, so that

$$\Delta=\begin{vmatrix} 0, & a_{12}, & a_{13},\ldots\ldots, & a_{1p}\\ a_{21}, & 0, & a_{23},\ldots\ldots, & a_{2p}\\ a_{31}, & a_{32}, & 0,\ldots\ldots, & a_{3p}\\ \ldots & \ldots & \ldots & \ldots\\ a_{p1}, & a_{p2}, & a_{p3},\ldots\ldots, & 0 \end{vmatrix}\ldots\ldots\ldots\ldots\ldots\ldots(11);$$

then the solution of the equations (10) is of the form

$$\Delta y_r=V_r\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12),$$

where V_r is the value of the determinant obtained by replacing the column $a_{1,r}, a_{2,r}, a_{3,r},\ldots$ by $X_1, X_2, X_3, \ldots$; and r has the values $1, 2, 3, \ldots, p$.

57. So far as regards the effectiveness of the result, there are three alternatives to be considered.

First, if Δ do not vanish, the values of $y_1, y_2, \ldots, y_p$ in equations (12) are unique and determinate. The system (10) is then a consistent system and its members are independent of one another.

Second, if Δ vanish and also all the quantities V_r vanish, the values of $y_1, y_2, \ldots, y_p$ in equations (12) are (in their present form) indeterminate. The members of the system (10) may then be not independent of one another.

Third, if Δ vanish and if some of the quantities V_r do not vanish, the values of the corresponding quantities y_r are infinite and those of the remainder are indeterminate; if y_1 be infinite, so that μ is zero, it may happen that the values of y_r/y_1 are finite and determinate. The system (10) may or may not be a consistent system.

The primary discrimination among the alternatives is thus made by the value of Δ.

58. Now the determinant Δ defined by (11) has its constituents such that $a_{t,s} = -a_{s,t}$ and $a_{t,t} = 0$; hence it is a skew symmetrical determinant*.

If *p be an even integer*, the determinant Δ is a perfect square, which may vanish on account of the forms of $X_1, X_2, \ldots, X_p$, but will not necessarily do so.

If *p be an odd integer*, the determinant Δ is evanescent whatever be the quantities $X_1, X_2, \ldots, X_p$.

59. Taking the former of these two cases, let p be an *even* integer; and suppose in the first place that Δ *does not vanish*.

Let $$\Delta = P_p^2 = P^2 \qquad \text{(13)};$$

then P is a Pfaffian of order p and is determined by the laws

$$P_p = [1, 2, 3, \ldots, p]$$
$$= \sum_{s=2}^{p} a_{1,s} [s+1, s+2, \ldots, p, 2, \ldots, s-1],$$
$$P_2 = a_{12}.$$

If $A_{s,t}$ be the minor of $a_{s,t}$ in Δ, then

$$A_{s,t} = (-1)^{s+t} PP_{s,t}$$
$$= -A_{t,s} = (-1)^{s+t+1} PP_{t,s},$$

where $P_{i,k}$ is, for $i < k$, the Pfaffian obtained from P by the omission from its symbol $[1, 2, \ldots, p]$ of the integers i and k.

Also, if $s < r - 1$, then

$$P_{s,r} = [1, 2, \ldots, s-1, s+1, \ldots, r-1, r+1, \ldots, p]$$
$$= (-1)^{s-1} [s+1, s+2, \ldots, r-1, r+1, \ldots, p, 1, 2, \ldots, s-1];$$

if $s = r - 1$, then

$$P_{r-1,r} = [1, 2, \ldots, r-2, r+1, \ldots, p]$$
$$= (-1)^{r-2} [r+1, r+2, \ldots, p, 1, \ldots, r-2];$$

if $s = r + 1$, then

$$P_{r+1,r} = -P_{r,r+1}$$
$$= -(-1)^{r-1} [r+2, r+3, \ldots, p, 1, \ldots, r-1]$$

* Scott's *Determinants*, Chap. vi., §§ 4—16, where the properties of such determinants and their minors are proved.

(by the preceding case)

$$=(-1)^r[r+2, r+3, \dots, p, 1, \dots, r-1];$$

and, if $s > r+1$, then

$$\begin{aligned} P_{s,r} &= -P_{r,s} \\ &= -[1, 2, \dots, r-1, r+1, \dots, s-1, s+1, \dots, p] \\ &= -(-1)^{s-2}[s+1, s+2, \dots, p, 1, \dots, r-1, r+1, \dots, s-1]; \end{aligned}$$

so that *in all cases when p is even*

$$P_{s,r} = (-1)^{s-1}[a, b, \dots, k],$$

where $a, b, \dots, k$ are the integers $1, 2, \dots, p$ (with s and r omitted) taken in their cyclical order beginning with that integer next after s which remains.

Hence equations (12) become

$$\begin{aligned} P^2 y_r &= V_r \\ &= \sum_{s=1}^{p} X_s A_{s,r} \\ &= \sum_{s=1}^{p} (-1)^{s+r} P X_s P_{s,r}, \end{aligned}$$

and therefore

$$\left.\begin{aligned} (-1)^{r-1} P y_r &= \sum_{s=1}^{p} (-1)^{s-1} X_s P_{s,r} \\ &= \sum_{s=1}^{p} X_s [s+1, s+2, \dots, s-1] \\ &= W_r \end{aligned}\right\} \dots\dots (14),$$

where, for every term under the sign of summation on the right-hand side, the integers $s, s+1, s+2, \dots, s-1$ are the integers $1, 2, \dots, r-1, r+1, \dots, p$ in their cyclical order and, in particular, the coefficient of X_{r-1} is $[r+1, \dots, p, 1, \dots, r-2]$ and the coefficient of X_r is zero.

Thus, for $p = 4$, the equations are

$$\left.\begin{aligned} [1234]\,y_1 &= + X_2[34] + X_3[42] + X_4[23] \\ -[1234]\,y_2 &= X_1[34] + X_3[41] + X_4[13] \\ [1234]\,y_3 &= X_1[24] + X_2[41] + X_4[12] \\ -[1234]\,y_4 &= X_1[23] + X_2[31] + X_3[12] \end{aligned}\right\},$$

where

$$[lm] = a_{lm},$$
$$[1234] = a_{12}a_{34} + a_{13}a_{42} + a_{14}a_{23};$$

and, for $p = 6$, the equations are

$$\left.\begin{array}{l} [123456]\,y_1 = \qquad +[3456]\,X_2+[4562]\,X_3+[5623]\,X_4+[6234]\,X_5+[2345]\,X_6 \\ -[123456]\,y_2=[3456]\,X_1 \qquad +[4561]\,X_3+[5613]\,X_4+[6134]\,X_5+[1345]\,X_6 \\ [123456]\,y_3=[2456]\,X_1+[4561]\,X_2 \qquad +[5612]\,X_4+[6124]\,X_5+[1245]\,X_6 \\ -[123456]\,y_4=[2356]\,X_1+[3561]\,X_2+[5612]\,X_3 \qquad +[6123]\,X_5+[1235]\,X_6 \\ [123456]\,y_5=[2346]\,X_1+[3461]\,X_2+[4612]\,X_3+[6123]\,X_4 \qquad +[1234]\,X_6 \\ -[123456]\,y_6=[2345]\,X_1+[3451]\,X_2+[4512]\,X_3+[5123]\,X_4+[1234]\,X_5 \end{array}\right\}.$$

It may be remarked that *all the quantities W_r in equations* (14) *cannot vanish*; for otherwise, the determinant of the coefficients of the non-vanishing quantities X in W must vanish, but the determinant is equal to P^{p-2} (as is easily proved) and therefore, on the present hypothesis, not zero.

And similarly *all but one of these quantities W_r cannot vanish.* For forming the function

$$X_1 W_1 - X_2 W_2 + X_3 W_3 - \ldots\ldots - X_p W_p,$$

it vanishes identically and therefore, if all the quantities W_r save one, say W_1, vanish, then $X_1 W_1$ vanishes or, since X_1 is not zero, W_1 also vanishes; so that all the functions W would then vanish which, on the hypothesis of a non-vanishing Δ, has just been proved impossible.

60. It follows then that, when p is an even integer and Δ does not vanish, the equations (14) constitute a determinate solution of the equations (10). The quantities y are

$$y_p = \frac{1}{\mu}, \qquad y_r = \frac{1}{\mu}\frac{\partial x_r}{\partial x_p},$$

so that by (14)

$$\frac{1}{\mu}\frac{\partial x_r}{\partial x_p} = (-1)^{r-1}\frac{W_r}{P}, \qquad \frac{1}{\mu} = (-1)^{p-1}\frac{W_p}{P} = -\frac{W_p}{P},$$

and therefore

$$\frac{\partial x_r}{\partial x_p} = (-1)^r \frac{W_r}{W_p}.$$

Hence, to obtain the functions x_r of § 54, we have the $p-1$ ordinary differential equations

$$\frac{dx_1}{W_1} = \frac{dx_2}{-W_2} = \frac{dx_3}{W_3} = \ldots\ldots = \frac{dx_{p-1}}{W_{p-1}} = \frac{dx_p}{-W_p} \quad\ldots\ldots\ldots(15),$$

equations which are called the *subsidiary Pfaffian system.*

At least two of the quantities W do not vanish, one of which may be supposed to be W_p*; so that the equations of the subsidiary system determine $x_1, x_2, \ldots, x_{p-1}$ as functions of x_p, the functions involving arbitrary constants. Thus we might have the integrals of (15) in the form

$$I_1 = a_1,\ I_2 = a_2, \ldots\ldots, I_{p-1} = a_{p-1},$$

where $I_1, I_2, \ldots, I_{p-1}$ are independent functions of $x_1, x_2, \ldots, x_p$, and where $a_1, a_2, \ldots, a_{p-1}$ are arbitrary independent quantities, constant so far as variations of x_p are concerned.

Now the equations of substitution in § 54 imply that $x_1, x_2, \ldots, x_{p-1}$ are to be functions of $u_1, u_2, \ldots, u_{p-1}$ as well as of x_p; while the integral equations just obtained involve only one, viz. x_p, of these variables. The reason is that the subsidiary equations which give $x_1, x_2, \ldots, x_{p-1}$ do not involve any variations of these quantities dependent upon the variations of $u_1, u_2, \ldots, u_{p-1}$. Thus the variables u stand in the same relation to the equations as the quantities a; and therefore the subsidiary system will still be satisfied, if $a_1, a_2, \ldots, a_{p-1}$ be replaced by $p-1$ independent functions of $u_1, u_2, \ldots, u_{p-1}$, for example, by $u_1, u_2, \ldots, u_{p-1}$.

When, then, the equations thus obtained are used as giving substitutions for $x_1, x_2, \ldots, x_{p-1}$, the element dx_p is removed from the differential equation; and when we divide out by the factor M, that is, by $exp \int (-P \div W_p)\, dx_p$, the variable x_p is also removed. The differential equation then takes the form

$$\Omega_1 = U_1 du_1 + U_2 du_2 + \ldots\ldots + U_{p-1} du_{p-1} = 0,$$

where $U_1, U_2, \ldots, U_{p-1}$ are functions of $u_1, u_2, \ldots, u_{p-1}$ alone.

Hence we can enunciate the theorem:—

When the coefficients of the total differential equation

$$\Omega = X_1 dx_1 + X_2 dx_2 + \ldots\ldots + X_{2n} dx_{2n} = 0$$

* The subsidiary system is symmetrical in all the variables of the original equation; so that, if W_p were found to vanish and W_q did not vanish, the desired reduction of the differential equation would be effected by the removal of x_q and dx_q, instead of x_p and dx_p, from explicit occurrence.

are functions of $x_1, x_2, \ldots, x_{2n}$, *such that the determinant* Δ, *with constituents*

$$a_{ij} = \frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i} \quad (i, j = 1, 2, \ldots, 2n),$$

does not vanish, then the differential equation can be transformed into

$$\Omega_1 = U_1 du_1 + U_2 du_2 + \ldots\ldots + U_{2n-1} du_{2n-1} = 0,$$

where $U_1, U_2, \ldots, U_{2n-1}$ *are functions of* $u_1, u_2, \ldots, u_{2n-1}$ *alone, by substitutions*

$$u_1 = I_1,\ u_2 = I_2,\ \ldots\ldots,\ u_{2n-1} = I_{2n-1}.$$

The quantities $I_1, I_2, \ldots, I_{2n-1}$ *are functions of the variables* $x_1, x_2, \ldots, x_{2n-1}, x_{2n}$, *determined so that*

$$I_1 = a_1,\ I_2 = a_2,\ \ldots\ldots,\ I_{2n-1} = a_{2n-1}$$

are $2n-1$ *independent integrals of the subsidiary Pfaffian system* (15) *associated with the original equation* $\Omega = 0$, *viz., of*

$$\frac{dx_1}{W_1} = \frac{dx_2}{-W_2} = \frac{dx_3}{W_3} = \ldots\ldots = \frac{dx_{2n-1}}{W_{2n-1}} = \frac{dx_{2n}}{-W_{2n}} \quad \ldots\ldots\ldots\ldots(15).$$

And the relation between the quantities Ω *and* Ω_1 *is*

$$\Omega = \Omega_1 \, exp \int (-P \div W_{2n})\, dx_{2n},$$

where P, *the square root of* Δ, *is the Pfaffian constructed from the coefficients* $X_1, X_2, \ldots, X_{2n}$.

And a similar reduction by one unit of the number of differential elements in the differential equation can be made by means of substitutions

$$f_1 = I_1,\ f_2 = I_2,\ \ldots\ldots,\ f_{2n-1} = I_{2n-1},$$

where $I_1, I_2, \ldots, I_{2n-1}$ *are the same as before and* $f_1, f_2, \ldots, f_{2n-1}$ *are* $2n-1$ *independent (but otherwise arbitrary) functions of*

$$u_1, u_2, \ldots\ldots, u_{2n-1}.$$

Before passing to the next case, it may be remarked that a very simple symbolical form (due to Cayley) may be given to the subsidiary Pfaffian system. Replacing X_m by $a_{0,m}$ for each of the indices m, we have

$$W_r = \sum_{s=1}^{p} a_{0,s}[s+1, s+2, \ldots, s-1]$$
$$= [0, 1, 2, \ldots, p],$$

where in the series 0, 1, 2, ..., p the integer r is omitted; and therefore

$$(-1)^r W_r = [r+1, r+2, \ldots, p, 0, \ldots, r-1].$$

The subsidiary system (15) now, on the omission of a factor -1, takes the form

$$\frac{dx_1}{[2, 3, 4, \ldots, p, 0]} = \frac{dx_2}{[3, 4, \ldots, p, 0, 1]} = \frac{dx_3}{[4, 5, \ldots, p, 0, 1, 2]} = \ldots\ldots$$
$$= \frac{dx_p}{[0, 1, 2, \ldots, p-1]},$$

in the general case; and, in particular for $p = 4$, the equations have the form

$$\frac{dx_1}{[2340]} = \frac{dx_2}{[3401]} = \frac{dx_3}{[4012]} = \frac{dx_4}{[0123]}.$$

61. Next, let p still be an even integer, but suppose now that Δ *does vanish*; so that

$$\rho = \sqrt{\Delta} = \sum_{s=2}^{p} a_{1,s}[s+1, s+2, \ldots, p, 2, \ldots, s-1] = 0,$$

with other equivalent summations according to the general laws in § 59.

Two cases require discussion (i) that in which not all the Pfaffians of order $p-2$, composed of the quantities a_{ij}, ~~do not~~ vanish, (ii) that in which they all do vanish.

62. For the former, it is necessary that at least two of the quantities W_r on the right-hand side of (14) do not vanish; for every Pfaffian of order $p-2$ occurs twice in the system of W's in complementary positions, that is, if it occur as the coefficient of X_i in W_j then (save as to sign) it occurs as the coefficient of X_j in W_i. Suppose that W_p is one of the non-vanishing W's; if it were zero, the only difference would be that we should attempt to eliminate from explicit occurrence in Ω some other variable with the same index as one of the non-vanishing W's.

In Ω let X_r be increased by an arbitrary variable quantity λ_r; the quantities a_{ij} are unaltered unless i or j is r and then

$$a'_{rs} = a_{rs} + \frac{\partial \lambda_r}{\partial x_s},$$

so that

$$\begin{aligned}
\Delta' &= \sum_{s=1}^{p} a'_{rs} [s+1, s+2, \ldots, p, 1, \ldots, s-1] \\
&= \sum_{s=1}^{p} \left(a_{rs} + \frac{\partial \lambda}{\partial x_s}\right) [s+1, s+2, \ldots, p, 1, \ldots, s-1] \\
&= \sum_{s=1}^{p} \frac{\partial \lambda_r}{\partial x_s} [s+1, s+2, \ldots, p, 1, \ldots, s-1],
\end{aligned}$$

because Δ is zero. Since λ_r is an arbitrary quantity, Δ' can only vanish, if the coefficients of the derivatives of λ_r all vanish, being those Pfaffians of order $p-2$ which occur in W_r. Two at least of the quantities W in (14) do not vanish, being W_p and at least one other which may be taken as W_r.

We thus have a new system of equations similar to (14) of the form

$$(-1)^{n-1} P' y_n' = W_n';$$

and they give

$$\left(\frac{\partial x_n}{\partial x_p}\right)' = (-1)^n \frac{W_n'}{W_p'} \quad (n = 1, 2, \ldots, p-1),$$

$$\frac{1}{\mu'} = -\frac{W_p'}{P'}.$$

Now let us pass to the former equations by making the arbitrary quantity λ zero; the $p-1$ equations giving the new values of $\left(\frac{\partial x_n}{\partial x_p}\right)'$ come to be the same as (15), which therefore are still valid under the present circumstances. But the modified form of P' is P the value of which is zero, and $W_{p'}$ comes to be W_p and is not zero; hence the modified form of μ', being the original μ, is zero and therefore (§ 54)

$$\frac{1}{M} \frac{\partial M}{\partial x_p} = 0,$$

so that M is independent of x_p. Considering then the origin of M in § 54, it follows that M *is either a constant or a function of* $x_2, \ldots, x_p$.

Hence we infer:—

The theorem of § 60 is still true when the determinant Δ vanishes and the equations (15) are a subsidiary system valid for the desired reduction, provided at least two of the quantities W do not vanish; and the effect of the vanishing of Δ is to transform Ω into Ω_1 without the necessity for the removal of any integral factor involving x_p, for such a factor cannot then occur.

If some of the quantities W_n in the subsidiary equations vanish, only formal mention need be made of the evident result—that the corresponding variables in Ω need not be transformed: for in such a case an integral of the subsidiary system is given by

$$u_n = x_n.$$

Ex. 1. The equation

$$\Omega = x_2 dx_1 + x_3 dx_2 + x_1 dx_3 + x_4 dx_4 = 0$$

given by Monge (l. c. § 45, p. 533; an integral equivalent of which is formed by him in three equations) falls under this head; for Δ is zero and the subsidiary system is

$$\frac{dx_1}{x_4} = \frac{dx_2}{x_4} = \frac{dx_3}{x_4} = \frac{dx_4}{x_1 + x_2 + x_3}.$$

Integrals of this system are

$$u_1 = x_1 - x_2,$$
$$u_2 = x_1 - x_3,$$
$$u_3 = 3x_4^2 - (x_1 + x_2 + x_3)^2;$$

and then $$\Omega = \tfrac{1}{3}(2u_2 - u_1)\, du_1 - \tfrac{1}{3}(u_1 + u_2)\, du_2 + \tfrac{1}{6} du_3.$$

Ex. 2. Similarly for the equation

$$\Omega = 0 = \{(x_1 + a) s + x_1^2\}\, dx_1 + \{(x_2 + a) s + x_2^2\}\, dx_2 + \{(x_3 + a) s + x_3^2\}\, dx_3 + \{(x_4 + a) s + x_4^2\}\, dx_4,$$

in which a is a constant and s denotes $x_1 + x_2 + x_3 + x_4$; it will be found that Δ vanishes, that the subsidiary system has three integrals which may be taken

$$u_1 = x_1 + x_2 + x_3 + x_4,$$
$$u_2 = x_1^2 + x_2^2 + x_3^2 + x_4^2,$$
$$u_3 = x_1^3 + x_2^3 + x_3^3 + x_4^3,$$

and that $$\Omega = \tfrac{1}{2} u_1\, du_2 + a u_1\, du_1 + \tfrac{1}{3} du_3.$$

Ex. 3. It may occur that Δ vanishes through having some row of its constituents all zero; for instance, $a_{12}, a_{13}, \ldots, a_{1,p}$ may all vanish. These give the relations

$$\frac{\partial X_1}{\partial x_r} = \frac{\partial X_r}{\partial x_1} \quad (r = 2, 3, \ldots\ldots, p).$$

Hence there exists a function u such that

$$X_1 = \frac{\partial u}{\partial x_1},$$

$$X_r = \frac{\partial u}{\partial x_r} + v_r,$$

where v_r is independent of x_1 but may be a function of $x_2, x_3, \ldots, x_p$; and then

$$\Omega = du + v_2 dx_2 + v_3 dx_3 + \ldots\ldots + v_p dx_p,$$

the desired reduction being thus effected and the multiplier M being unity.

63. For the second case, in which all the Pfaffians of order $p-2$ vanish as well as Δ, there must exist some lower limit of order at which the Pfaffians do not all vanish; for it is assumed that the quantities $a_{s,t}$ do not all vanish. Let this order be m, and suppose that $[1, 2, \ldots, m]$, $[2, 3, \ldots, m+1]$, and others of this order do not vanish; but that all the Pfaffians of the system of orders $m+2$ and higher (even) integers vanish.

The result of § 62 may be applied. Let $m+1$ new variables $u_2, u_3, \ldots, u_{m+2}$ be introduced, which are functions of $x_1, x_2, \ldots, x_p$, such that when we substitute for $x_2, \ldots, x_{m+2}$ in Ω, the term in dx_1 is absent; then we have

$$\Omega = M(U_2 du_2 + U_3 du_3 + \ldots + U_{m+2} du_{m+2}) + \sum_{m+3}^{r=p} Y_r dx_r,$$

and the quantity M is independent of x_1. It will now be shewn that Y_r, $(r = m+3, \ldots, p)$, is *explicitly independent* of x_1, so that it is a function of $u_2, \ldots, u_{m+2}, x_{m+3}, \ldots, x_p$ only.

By comparing the two expressions for Ω we have

$$Y_r = X_r + \sum_{s=2}^{m+2} X_s \frac{\partial x_s}{\partial x_r}$$

for the values $m+3, \ldots, p$ of r; hence

$$\begin{aligned} \frac{\partial Y_r}{\partial x_1} = \frac{\partial X_r}{\partial x_1} &+ \sum_{s=2}^{m+2} \frac{\partial X_r}{\partial x_s} \frac{\partial x_s}{\partial x_1} \\ &+ \sum_{s=2}^{m+2} \frac{\partial x_s}{\partial x_r} \left(\frac{\partial X_s}{\partial x_1} + \frac{\partial X_s}{\partial x_2} \frac{\partial x_2}{\partial x_1} + \ldots\ldots + \frac{\partial X_s}{\partial x_{m+2}} \frac{\partial x_{m+2}}{\partial x_1} \right) \\ &+ \sum_{s=2}^{m+2} X_s \frac{\partial^2 x_s}{\partial x_1 \partial x_r}. \end{aligned}$$

But since dx_1 is absent from the second expression for Ω, we have

$$0 = X_1 + X_2 \frac{\partial x_2}{\partial x_1} + \dots\dots + X_{m+2} \frac{\partial x_{m+2}}{\partial x_1}.$$

Taking now the complete derivative of both sides with regard to x_r and subtracting from the value of $\frac{\partial Y_r}{\partial x_1}$ the right-hand side of the derived equation, it is easy to find (after an arrangement of terms similar to that in § 55) the equation

$$\frac{\partial Y_r}{\partial x_1} = a_{r,1} + a_{r,2} \frac{\partial x_2}{\partial x_1} + \dots\dots + a_{r,m+2} \frac{\partial x_{m+2}}{\partial x_1}$$

$$+ \sum_{s=2}^{m+2} \frac{\partial x_s}{\partial x_r} \left(a_{s,1} + a_{s,2} \frac{\partial x_2}{\partial x_1} + \dots\dots + a_{s,m+2} \frac{\partial x_{m+2}}{\partial x_1} \right).$$

But, by §§ 60, 62, the values of the functions $x_2, \dots, x_{m+2}$ are determined by equations of the form

$$\frac{dx_1}{W_1} = \frac{dx_2}{-W_2} = \frac{dx_3}{W_3} = \dots\dots = \frac{dx_{m+2}}{-W_{m+2}},$$

where the quantities W are of the form

$$W_r = \sum_{s=1}^{m+2} X_s [s+1, s+2, \dots, s-1]$$

defined as in § 59; and, as in the other cases, W_1 does not vanish, while the quantity μ for the present case does vanish.

Now a reference to the equations for the particular case, which are the same as (8) in the general case save that, as just remarked, μ vanishes, shews that the coefficient of each of the terms $\frac{\partial x_s}{\partial x_r}$ $(s = 2, \dots, m+2)$ in $\frac{\partial Y_r}{\partial x_1}$ vanishes, so that

$$\frac{\partial Y_r}{\partial x_1} = a_{r1} + a_{r2} \frac{\partial x_2}{\partial x_1} + \dots\dots + a_{r,m+2} \frac{\partial x_{m+2}}{\partial x_1}.$$

When in the right-hand side we substitute for the various terms $\frac{\partial x_2}{\partial x_1}, \frac{\partial x_3}{\partial x_1}, \dots$, their values in terms of the quantities W, it is easily proved by the properties of Pfaffians that

$$W_1 \frac{\partial Y_r}{\partial x_1} = \sum_{s=2}^{m+2} X_s [1, 2, 3, \dots, s-1, r, s+1, \dots, m+2].$$

But the coefficients of X_s on the right-hand side are all of them Pfaffians of order $m+2$, which vanish by our initial hypothesis; hence, as W_1 does not vanish, we have $\frac{\partial Y_r}{\partial x_1} = 0$, or the coefficients Y_r are explicitly independent of the variable x_1 when the substitutions are effected.

Hence *when all the Pfaffians of the system of order higher than* m *vanish, it is sufficient to use equations for* $m+2$ *variables similar to those of* § 62 *and so to transform the first* $m+2$ *terms of the differential expression into* $m+1$ *terms by the introduction of new variables determined from those equations; when the transformation by means of these variables is effected throughout the expression* Ω*, all the coefficients are explicitly independent of the variable, the differential element of which has been removed from explicit occurrence**.

64. All the alternatives, in the case when p is an even integer, have now been discussed, with the following general result:—

"A differential expression Ω containing an even number of "differential elements can always be transformed into another, "which contains the next smaller odd number of differential "elements; the new variables which are necessary for this trans-"formation are determined as in §§ 60, 62, 63 according to the "properties of the coefficients in Ω; and the new coefficients are "such that, except in the form of a (possible) common factor, "they involve only the (smaller number of) new variables."

This transformation, by which the even number of differential elements is reduced by unity, may be called the *even reduction;* and it is to be borne in mind that, by this even reduction, the number of variables, which occur in the new coefficients (save in the common factor), is also reduced by unity.

Also, this even reduction is not unique. For the purpose of the transformation certain new variables are introduced, being derived from the integration of certain differential equations.

* The effect of the vanishing of Δ and its minors upon the form of the normal reduced equivalent of Ω will be discussed later (§§ 117 et seq., §§ 144 et seq.): the object of the present chapter is to prove the possibility of the reduction on the lines of Pfaff's original memoir.

Though all solutions of these differential equations are functionally equivalent to one another, the forms may be different; and thus we may have different sets of new variables introduced, each set leading to an even reduction, with some kind of functional relation among the sets.

65. In the case in which p is an odd integer, Δ vanishes identically whatever be the values of the quantities X; so that the process adopted in § 61, whereby a temporarily non-evanescent determinant was formed, is no longer effective.

Now, in general, the first minors of Δ do not vanish, so that the values of the quantities y_r are infinite. Instead of taking these infinite values, which are subsidiary to the determination of the values of $\frac{\partial x_r}{\partial x_p}$, we return to the equations (4) and (8), retaining all but the last of (8) which is (4) transformed after multiplication by μ. If μ be not zero, the last of (8) may be retained; if μ be zero, we use (4) in place of that last equation.

Taking these cases in order we have first, from the full set (8), the relation

$$\mu \sum_{r=1}^{p} X_r A_{r,p} = 0,$$

where $A_{r,p}$ is the minor of $a_{r,p}$ in Δ; and therefore since μ is not zero

$$\sum_{r=1}^{p} X_r A_{r,p} = 0,$$

or substituting for $A_{r,p}$ its value*, we have

$$\sum_{r=1}^{p} X_r [r+1, r+2, \ldots, r-1] = 0.$$

Secondly, if μ be zero, we solve the first $p-1$ equations of (8) and substitute the values of $\frac{\partial x_r}{\partial x_p}$ thence derived in (4); and multiplying up by $[1, 2, 3, \ldots, p-1]$, which is not in general zero, we come to the condition obtained in the earlier case.

* Scott's *Determinants*, § 14; and § 59 preceding.

It therefore follows that, in order to have the subsidiary equations a consistent system, the condition

$$\sum_{r=1}^{p} X_r [r+1, r+2, \ldots, r-1] = 0$$

must be satisfied.

66. If *this condition be satisfied*, the system contains only $p-1$ independent equations at the utmost; and they are not sufficient to determine μ and the $p-1$ derivatives with regard to x_p.

To transform $\Omega = 0$ into an equation, so that it may contain one variable fewer and be without the differential element of that variable, we proceed as in § 63. Neglecting for the moment the variation of x_1, we transform

$$X_2 dx_2 + X_3 dx_3 + \ldots\ldots + X_p dx_p$$

into $$M(U_2 du_2 + U_3 du_3 + \ldots\ldots + U_{p-1} du_{p-1}),$$

by equations

$$\mu X_2 = a_{2p} + \qquad a_{23}\frac{\partial x_3}{\partial x_p} + a_{24}\frac{\partial x_4}{\partial x_p} + \ldots\ldots + a_{2,p-1}\frac{\partial x_{p-1}}{\partial x_p},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\mu X_p = \qquad a_{p2}\frac{\partial x_2}{\partial x_p} + a_{p3}\frac{\partial x_3}{\partial x_p} + \ldots\ldots \qquad + a_{p,p-1}\frac{\partial x_{p-1}}{\partial x_p}.$$

Then if

$$\Omega = X_1 dx_1 + X_2 dx_2 + \ldots\ldots \qquad + X_p dx_p$$
$$= M(U_2 du_2 + U_3 du_3 + \ldots\ldots + U_{p-1} du_{p-1}) + Y_1 dx_1,$$

when we resume consideration of the variation of x_1, we have

$$Y_1 = X_1 + X_2\frac{\partial x_2}{\partial x_1} + X_3\frac{\partial x_3}{\partial x_1} + \ldots\ldots + X_{p-1}\frac{\partial x_{p-1}}{\partial x_1};$$

and also

$$0 = X_p + X_2\frac{\partial x_2}{\partial x_p} + X_3\frac{\partial x_3}{\partial x_p} + \ldots\ldots + X_{p-1}\frac{\partial x_{p-1}}{\partial x_p}.$$

From the former we have, as in § 63,

$$\frac{\partial Y_1}{\partial x_p} = \frac{\partial X_1}{\partial x_p} + \sum_{s=2}^{p-1}\frac{\partial X_1}{\partial x_s}\frac{\partial x_s}{\partial x_p} + \sum_{i=2}^{p-1} X_i \frac{\partial^2 x_i}{\partial x_1 \partial x_p}$$
$$+ \sum_{i=2}^{p-1}\frac{\partial x_i}{\partial x_1}\left(\frac{\partial X_i}{\partial x_p} + \frac{\partial X_i}{\partial x_2}\frac{\partial x_2}{\partial x_p} + \ldots\ldots + \frac{\partial X_i}{\partial x_{p-1}}\frac{\partial x_{p-1}}{\partial x_p}\right);$$

and from the latter

$$0 = \frac{\partial X_p}{\partial x_1} + \sum_{i=2}^{p-1} \frac{\partial X_p}{\partial x_i} \frac{\partial x_i}{\partial x_1} + \sum_{i=2}^{p-1} X_i \frac{\partial^2 x_i}{\partial x_1 \partial x_p}$$

$$+ \sum_{s=2}^{p-1} \frac{\partial x_s}{\partial x_p} \left(\frac{\partial X_s}{\partial x_1} + \frac{\partial X_s}{\partial x_2} \frac{\partial x_2}{\partial x_1} + \ldots\ldots + \frac{\partial X_s}{\partial x_{p-1}} \frac{\partial x_{p-1}}{\partial x_1} \right).$$

Subtracting corresponding sides of the latter from the former equation, it follows that

$$\frac{\partial Y_1}{\partial x_p} = a_{1p} + \sum_{i=2}^{p-1} a_{1i} \frac{\partial x_i}{\partial x_p} + \sum_{s=2}^{p-1} \frac{\partial x_s}{\partial x_1} \left\{ a_{sp} + \sum_{j=2}^{p-1} a_{sj} \frac{\partial x_j}{\partial x_p} \right\}.$$

The coefficient of $\frac{\partial x_s}{\partial x_1}$ on the right-hand side is μX_s; let

$$\Theta = a_{1p} + a_{2p} \frac{\partial x_2}{\partial x_p} + a_{3p} \frac{\partial x_3}{\partial x_p} + \ldots\ldots + a_{p-1,p} \frac{\partial x_{p-1}}{\partial x_p},$$

so that

$$\frac{\partial Y_1}{\partial x_p} = \Theta + \mu \sum_{s=2}^{p-1} X_s \frac{\partial x_s}{\partial x_1}.$$

Substituting from the subsidiary equations, used above in the preliminary transformation, for $\frac{\partial x_2}{\partial x_p}, \ldots, \frac{\partial x_{p-1}}{\partial x_p}$, we have

$$\begin{vmatrix} \Theta, & a_{12}, & a_{13}, & \ldots\ldots, & a_{1p} \\ \mu X_2, & 0, & a_{23}, & \ldots\ldots, & a_{2p} \\ \mu X_3, & a_{32}, & 0, & \ldots\ldots, & a_{3p} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \mu X_p, & a_{p2}, & a_{p3}, & \ldots\ldots, & 0 \end{vmatrix} = 0.$$

But the condition expressed at the end of § 65, supposed to be satisfied in the present case, may be written

$$\begin{vmatrix} \mu X_1, & a_{12}, & a_{13}, & \ldots\ldots, & a_{1p} \\ \mu X_2, & 0, & a_{23}, & \ldots\ldots, & a_{2p} \\ \mu X_3, & a_{32}, & 0, & \ldots\ldots, & a_{3p} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \mu X_p, & a_{p2}, & a_{p3}, & \ldots\ldots, & 0 \end{vmatrix} = 0.$$

Hence $\Theta = \mu X_1$; and therefore

$$\frac{\partial Y_1}{\partial x_p} = \mu \left\{ X_1 + \sum_{s=2}^{p-1} X_s \frac{\partial x_s}{\partial x_1} \right\} = \mu Y_1,$$

which is the requisite condition that, if x_p occur at all in Y_1, it

occurs only in a factor M; and therefore Ω is transformed to

$$M(U_2du_2 + \dots\dots + U_{p-1}du_{p-1} + Z_1dx_1),$$

where $Y_1 = MZ_1$ and Z_1 is explicitly independent of x_p. The differential expression is thus transformed in the required manner.

Ex. Let $p=3$ and suppose the condition satisfied, viz.

$$X_1[23]+X_2[31]+X_3[12]=0,$$

the known condition of integrability of the equation

$$X_1dx_1+X_2dx_2+X_3dx_3=0.$$

The equations in their most general form taken so as to remove x_3 are

$$\mu X_1 = a_{13}+a_{12}\frac{\partial x_2}{\partial x_3},$$

$$\mu X_2 = a_{23}+a_{21}\frac{\partial x_1}{\partial x_3},$$

$$0 = X_3+X_1\frac{\partial x_1}{\partial x_3}+X_2\frac{\partial x_2}{\partial x_3}.$$

If we take $\mu=0$ so that the third equation is satisfied in virtue of the first two, the new variables are obtained by integrating the equations

$$\frac{dx_1}{a_{23}}=\frac{dx_2}{a_{31}}=\frac{dx_3}{a_{12}},$$

and this, in effect, is Bertrand's method (§ 16).

If, however, we do not take μ zero, we may omit the first of the three equations (for only two of them are independent) and use the second to determine μ; there is then left only a single equation

$$0=X_3+X_1\frac{\partial x_1}{\partial x_3}+X_2\frac{\partial x_2}{\partial x_3}$$

to be satisfied. We may then assign any condition, not inconsistent with this equation, for the determination of x_1 and x_2. Thus as a condition we may require that x_1 shall be unchanged; then the original variable x_1 and the new variable, which is obtained from the integration of

$$0=X_3+X_2\frac{\partial x_2}{\partial x_3},$$

will be the new variables leading to the required transformation. This is, in effect, the ordinary (Euler's) method of integration of the equation.

As a special instance, consider the equation

$$\Omega=(cx_2-bx_3)\,dx_1+(ax_3-cx_1)\,dx_2+(bx_1-ax_2)\,dx_3=0.$$

We have $a_{12}=2c$, $a_{23}=2a$, $a_{31}=2b$; the subsidiary equations are (omitting a factor $\frac{1}{2}$), in the first method,

$$\frac{dx_1}{a}=\frac{dx_2}{b}=\frac{dx_3}{c},$$

so that new variables are

$$u=\frac{x_1}{a}-\frac{x_2}{b}, \qquad v=\frac{x_2}{b}-\frac{x_3}{c};$$

and it is easily shewn that

$$\Omega=abc(vdu-udv),$$

so that $M=abc$ and $\mu=0$.

Proceeding by the second method and keeping x_1 unchanged we have, as the equation determining the new variable,

$$(ax_3-cx_1)\frac{dx_2}{dx_3}+(bx_1-ax_2)=0,$$

so that we take x_1 and $\qquad u=\dfrac{bx_1-ax_2}{ax_3-cx_1}$

as the new variables. It is easily shewn that

$$\Omega=-\frac{(ax_3-cx_1)^2}{a}du,$$

so that $M=-(ax_3-cx_1)^2/a$ and $\mu\left(=\dfrac{1}{M}\dfrac{\partial M}{\partial x_3}\right)$ is not zero.

67. If the condition of § 65 for the coexistence of the determining equations be *not satisfied*, then the system of those equations is not consistent and therefore the transformation, which is to be determined by them, is not possible. Hence the most general equation, which contains an odd number of differential elements, cannot be subjected to a complete even reduction, that is, the number of differential elements cannot be reduced by one unit so as to leave the coefficients of those elements functions of the (reduced number of) new variables, save as to a common factor.

But though this reduction be not possible, another transformation may be effected. Let the first $2n-2$ terms of

$$\Omega=X_1dx_1+X_2dx_2+\ldots\ldots+X_{2n-2}dx_{2n-2}+X_{2n-1}dx_{2n-1}$$

be denoted by Φ, which contains an even number of differential elements, and may contain all the variables. Let an even reduction be applied to Φ on the supposition that x_{2n-1} does not change, so that the only variables which are transformed

are $x_1, x_2, \ldots, x_{2n-2}$; then Φ comes to be of the form $M_1\Phi'$, where Φ' denotes

$$U_1 du_1 + U_2 du_2 + \ldots\ldots + U_{2n-3} du_{2n-3}.$$

Of the quantities which occur in Φ', the variables $u_1, u_2, \ldots, u_{2n-3}$ are functions of $x_1, x_2, \ldots, x_{2n-2}$ and of any non-varying quantities that may occur in $X_1, X_2, \ldots, X_{2n-2}$, i.e. of x_{2n-1} also in general; the coefficients $U_1, U_2, \ldots, U_{2n-3}$ are explicit functions of $u_1, u_2, \ldots, u_{2n-3}$ and of any non-varying quantities that may occur in $X_1, X_2, \ldots, X_{2n-2}$, i.e. of x_{2n-1} also in general. And M_1 is, in general, a function of all the variables in Ω.

Removing now the supposition of non-variability of x_{2n-1}, introduced for the purpose of changing the form of Φ, and taking the complete variation of that changed form, we have

$$\Omega = M_1 (U_1 du_1 + U_2 du_2 + \ldots\ldots + U_{2n-3} du_{2n-3}) + X'_{2n-1}\, dx_{2n-1},$$

where

$$X'_{2n-1} = X_{2n-1} - M_1 \left(U_1 \frac{\partial u_1}{\partial x_{2n-1}} + U_2 \frac{\partial u_2}{\partial x_{2n-1}} + \ldots\ldots + U_{2n-3} \frac{\partial u_{2n-3}}{\partial x_{2n-1}} \right).$$

Let this be denoted by

$$\Omega = M_1 \Omega_1 + X'_{2n-1} dx_{2n-1}.$$

The transformation thus effected is an incomplete even reduction, being applied to Φ only; in view of the fact that Ω, the quantity to be transformed, contains an odd number of variables the transformation may be called the ***odd reduction.*** It is a reduction which diminishes the number of differential elements by unity; but, unlike the even reduction, it does not in general leave the new coefficients, save as to a factor, in the form of explicit functions of the new variables alone.

Since the even reduction can always be applied to a quantity with an even number of differential elements such as Φ, it follows that a quantity Ω with an odd number of differential elements can always be subjected to an odd reduction.

And, as before, an odd reduction is not unique.

Ex. To form the odd reduction of

$$\Omega = x_5 dx_1 + x_1 dx_2 + x_2 dx_3 + x_3 dx_4 + x_4 dx_5.$$

Taking the first four terms and constructing the subsidiary equations of § 59 for the incomplete even reduction, we find them to be

$$\frac{dx_1}{x_1+x_3}=\frac{dx_2}{-x_5}=\frac{dx_3}{x_3}=\frac{dx_4}{-x_2-x_5}.$$

Taking $\theta=\log x_3$, these become

$$\frac{dx_1}{d\theta}=x_1+e^\theta,\quad \frac{dx_2}{d\theta}=-x_5,\quad \frac{dx_4}{d\theta}=-x_2-x_5;$$

and therefore new variables (§ 60) are given by

$$x_1=(u_1+\theta)e^\theta,$$
$$x_2=u_2-x_5\theta,$$
$$x_4=u_3-u_2\theta+\tfrac{1}{2}x_5(\theta-1)^2.$$

Passing now to the complete variations, so as to obtain the odd reduction, it is easy to shew that

$$\Omega=x_3(x_5du_1+u_1du_2+du_3)+X'_5dx_5,$$

where $$X'_5=u_3-u_2\theta+\tfrac{1}{2}x_5(\theta-1)^2+\tfrac{1}{2}e^\theta(1-2u_1\theta-2\theta-\theta^2).$$

This verifies the general proposition for the particular case; we have

$$M_1=x_3,\qquad \Omega_1=x_5du_1+u_1du_2+du_3.$$

68. We are now in a position to reduce the number of terms in any given differential expression to a general minimum. In particular cases the number of terms in the finally reduced form may be smaller than that expected from the number in this general minimum; the possibility of such a result depends upon the satisfaction of certain conditions, the consideration of which will be deferred for the present.

First, taking the case in which the expression Ω to be reduced contains an odd number $2n-1$ of differential elements, say

$$\Omega=X_1dx_1+X_2dx_2+\ldots\ldots+X_{2n-2}dx_{2n-2}+X_{2n-1}dx_{2n-1},$$

we apply to Ω an odd reduction so that

$$\Omega=M_1\Omega_1+X'_{2n-1}dx_{2n-1}.$$

Now Ω_1 contains an odd number of differential elements $u_1, u_2, \ldots, u_{2n-3}$, and their coefficients are functions of these variables and (possibly) of x_{2n-1}; hence applying to Ω_1 an odd reduction we have

$$\Omega_1=M_2\Omega_2+U'_{2n-3}du_{2n-3}+Y_{2n-1}dx_{2n-1},$$

where $$\Omega_2=V_1dv_1+V_2dv_2+\ldots\ldots+V_{2n-5}dv_{2n-5},$$

$$U'_{2n-3} = U_{2n-3} - M_2\left(V_1\frac{\partial v_1}{\partial u_{2n-3}} + \dots + V_{2n-5}\frac{\partial v_{2n-5}}{\partial u_{2n-3}}\right),$$

and

$$Y_{2n-1} = -M_2\left(V_1\frac{\partial v_1}{\partial x_{2n-1}} + \dots + V_{2n-5}\frac{\partial v_{2n-5}}{\partial x_{2n-1}}\right).$$

The variables $v_1, v_2, \dots, v_{2n-5}$ of Ω_2 are functions of $u_1, u_2, \dots, u_{2n-4}$, possibly of u_{2n-3} and possibly of x_{2n-1}, in case the latter occur in the coefficients $U_1, U_2, \dots, U_{2n-4}$; and the coefficients $V_1, V_2, \dots, V_{2n-5}$ of Ω_2 are functions of $v_1, v_2, \dots, v_{2n-5}$, possibly of u_{2n-3} and possibly of x_{2n-1}.

Ex. Thus the modification of Ω_1 in the last Example is

$$\Omega_1 = u_1 dv + du_3 - u_1 \log u_1\, dx_5,$$

where $v = u_2 + x_5 \log u_1$; and so

$$\Omega = x_3 u_1 dv + x_3 du_3 + (X'_5 - x_3 u_1 \log u_1)\, dx_5.$$

Applying now an odd reduction to Ω_2, which contains an odd number of differential elements, and bearing in mind the variables that occur in Ω_2, we obtain a result the most general form of which is

$$\Omega_2 = M_3\Omega_3 + V'_{2n-5}dv_{2n-5} + U''_{2n-3}du_{2n-3} + Z_{2n-1}dx_{2n-1},$$

where Ω_3 contains only $2n-7$ differential elements. Proceeding in this way, we shall ultimately reach a quantity Ω_{n-1} which contains only a single differential element, say of the form Pdp_1; and the value of Ω_{n-2}, which unreduced contains three differential elements, will be of the form

$$\Omega_{n-2} = M_{n-1}\Omega_{n-1} + Q'_3dq_3 + R'_5dr_5 + \dots + T_{2n-1}dx_{2n-1},$$

when all the variations are taken into account. Substituting then for all the quantities Ω_r in turn, we evidently have a result of the form

$$\Omega = P_1dp_1 + Q_3dq_3 + R_5dr_5 + \dots + W_{2n-1}dx_{2n-1};$$

for each unreduced Ω has two differential elements fewer than the Ω earlier in the succession, one of them having been removed by reduction and the other having been set aside in order to apply the incomplete even reduction.

The reduced form of Ω evidently contains n differential elements, that is, $\frac{1}{2}(p+1)$, where p is the odd number which

occurred in the original form of the expression; and the variables and the coefficients are, all of them, functions of the original variables.

Secondly, take the case in which the expression Ω to be reduced contains an even number $2n$ of differential elements, say

$$\Omega = Y_1 dy_1 + Y_2 dy_2 + \ldots + Y_{2n-1} dy_{2n-1} + Y_{2n} dy_{2n};$$

we apply to Ω an even reduction, so that

$$\Omega = M(X_1 dx_1 + X_2 dx_2 + \ldots + X_{2n-1} dx_{2n-1}) = M\Omega',$$

where Ω' contains only $2n - 1$ differential elements and the coefficients in Ω' are functions of the variables $x_1, x_2, \ldots, x_{2n-1}$. Thus Ω' is an expression of the type considered in the former case; carrying out the full reduction on Ω' as there indicated, we have a reduced form containing $\frac{1}{2}\{(2n-1)+1\}$, that is, n differential elements, and so

$$\begin{aligned}\Omega &= M(P_1 dp_1 + Q_3 dq_3 + \ldots + W_{2n-1} dx_{2n-1}) \\ &= P'_1 dp_1 + Q'_3 dq_3 + \ldots\ldots + W'_{2n-1} dx_{2n-1}.\end{aligned}$$

The reduced form of Ω contains n, that is, $\frac{1}{2}p$ differential elements, where p is the even number of differential elements which occurred in the original form of the expression.

It is to be noticed that, in the case when p is *even*, no one of the differential elements in the reduced form is the same as any one in the original form—always supposing Ω to be the most general expression possible—so that then P', p_1, Q'_3, q_3, $\ldots$, W'_{2n-1} and x_{2n-1} are together p functions of $y_1, y_2, \ldots, y_p$; and *one of the variables in the reduced form is an integral of the first subsidiary system.* But, in the case when p is *odd*, *one of the differential elements in the reduced form is the same as one in the original form*, so that then P_1, p_1, Q_3, q_3, $\ldots$ and W_{2n-1} are together p functions of $x_1, x_2, \ldots, x_p$.

We have now proved the possibility of reducing an expression containing p differential elements to one containing either $\frac{1}{2}p$ or $\frac{1}{2}(p+1)$ differential elements. Other and less laborious methods of effecting the reduction will be given, on the assumption of this now proved possibility; and they will take account of the fact, that the reduced form is not unique owing to the want of uniqueness in the successive steps of the reduction as effected by the foregoing method.

Ex. As a simple illustration of the general result we may deduce a theorem stated by Jacobi* relative to the equation

$$\Omega = X_1 dx_1 + X_2 dx_2 + X_3 dx_3 + X_4 dx_4 = 0.$$

The application to Ω of an even reduction gives

$$\Omega = M(U_1 du_1 + U_2 du_2 + U_3 du_3) = M\Omega',$$

where U_1, U_2, U_3 are functions of u_1, u_2, u_3 alone and M may be a function of x_1, x_2, x_3, x_4, while u_1, u_2, u_3 are determined from the integration of the equations (§ 60)

$$\frac{dx_1}{[2340]} = \frac{dx_2}{[3401]} = \frac{dx_3}{[4012]} = \frac{dx_4}{[0123]},$$

which, being three ordinary equations of the first degree, require three integrals in order to be satisfied, these three being taken (§ 60) to be u_1, u_2, u_3.

When an odd reduction is applied to Ω', one of the variables, say u_3, is left unchanged and we have a result of the form

$$\Omega' = N dv + V_3 du_3.$$

Hence the equation $\Omega = 0$ can be replaced by

$$N dv + V_3 du_3 = 0.$$

If now we make u_3 a constant, this equation comes to be $N dv = 0$, i.e. $dv = 0$, and so an integral equivalent of the equation is given by a single equation. Now the relation $u_3 =$ constant is one of the integrals of the subsidiary system; and thus it follows that, *when one of the integrals of the subsidiary system of the equation*

$$X_1 dx_1 + X_2 dx_2 + X_3 dx_3 + X_4 dx_4 = 0$$

is used to diminish by unity the number of differential elements and the number of variables, the resulting equation in only three differential elements can be represented by a single equation as its integral equivalent and therefore its coefficients are such as to satisfy the condition of integrability.

This is the theorem referred to. As a special instance, we may take Ex. 1, § 62. One of the integrals of the subsidiary system being

$$u_1 = x_1 - x_2,$$

we take

$$x_1 = x_2 + a,$$

where a is a constant; when this is used to transform the equation, the new form is

$$x_2 dx_2 + x_3 dx_2 + (x_2 + a)\, dx_3 + x_4 dx_4 = 0,$$

which is evidently integrable.

It may be added that the process thus indicated is one of the most effective processes for obtaining an integral equivalent of an unconditioned Pfaffian in four variables.

* *Crelle*, t. xxix., p. 253; see also Cayley, *Crelle*, t. lvii., pp. 273—277.

69. When the differential expression Ω has been reduced to the smallest number of terms, an integral equivalent of the equation $\Omega = 0$ is easily obtained.

(i) In the case of an even number of original variables, the final form of the differential equation is

$$U_1 du_1 + U_2 du_2 + \ldots\ldots + U_n du_n = 0,$$

where the quantities U, u are functions of the original variables, among which no identical relation subsists.

Then an integral equivalent, corresponding to the complete integral in partial differential equations of the first order, is given by

$$u_1 = a_1,\ u_2 = a_2, \ldots\ldots,\ u_n = a_n,$$

where $a_1, a_2, \ldots, a_n$ are constants; and, in virtue of these n equations, the equation $\Omega = 0$ is satisfied.

Another integral equivalent, corresponding to the general integral in partial differential equations of the first order, is given by

$$\left.\begin{array}{c} \phi(u_1, u_2, \ldots, u_n) = 0 \\ \dfrac{1}{U_1}\dfrac{\partial\phi}{\partial u_1} = \dfrac{1}{U_2}\dfrac{\partial\phi}{\partial u_2} = \ldots\ldots = \dfrac{1}{U_n}\dfrac{\partial\phi}{\partial u_n} \end{array}\right\},$$

where ϕ is any arbitrary function; again, the integral consists of n equations, i.e., of as many equations as there are differential elements in the reduced form. And if, instead of taking only a single arbitrary functional relation $\phi = 0$, we (less generally) take k such relations, say

$$\phi_1 = 0,\ \phi_2 = 0, \ldots\ldots,\ \phi_k = 0,$$

then there are other $n - k$ equations to be associated with these, being the $n - k$ equations which are the independent relations of the set

$$\left\| \begin{array}{cccc} \dfrac{\partial\phi_1}{\partial u_1}, & \dfrac{\partial\phi_1}{\partial u_2}, & \ldots\ldots\ldots\ldots, & \dfrac{\partial\phi_1}{\partial u_n} \\ \dfrac{\partial\phi_2}{\partial u_1}, & \dfrac{\partial\phi_2}{\partial u_2}, & \ldots\ldots\ldots\ldots, & \dfrac{\partial\phi_2}{\partial u_n} \\ \ldots & \ldots & \ldots & \ldots \\ \dfrac{\partial\phi_k}{\partial u_1}, & \dfrac{\partial\phi_k}{\partial u_2}, & \ldots\ldots\ldots\ldots, & \dfrac{\partial\phi_k}{\partial u_n} \\ U_1, & U_2, & \ldots\ldots\ldots\ldots, & U_n \end{array} \right\| = 0.$$

(ii) In the case of an odd number of original variables, the final form of the differential equation has been proved to be

$$U_1 dx + U_2 du_2 + \dots\dots + U_n du_n = 0,$$

where the quantities U and u are functions of the original variables one of which is x.

Then an integral equivalent, corresponding to the complete integral, would be given by taking

$$u_3 = a_3, \quad u_4 = a_4, \quad \dots\dots, \quad u_n = a_n$$

and (as it would not then be possible to integrate $U_1 dx + U_2 du_2 = 0$) by taking

$$\phi(x, u_2) = 0$$

with the relation

$$\frac{1}{U_1}\frac{\partial\phi}{\partial x} = \frac{1}{U_2}\frac{\partial\phi}{\partial u_2},$$

where ϕ is arbitrary. And another integral equivalent (really a more general case of the latter) is given by

$$\left.\begin{array}{c} \phi(x, u_2, \dots\dots, u_n) = 0 \\ \dfrac{1}{U_1}\dfrac{\partial\phi}{\partial x} = \dfrac{1}{U_2}\dfrac{\partial\phi}{\partial u_2} = \dots\dots = \dfrac{1}{U_n}\dfrac{\partial\phi}{\partial u_n} \end{array}\right\},$$

where ϕ is arbitrary.

It is, however, not necessary here to discuss the forms of solution, as the discussion will be an essential part of Clebsch's theory.

Ex. Reduce to its canonical form, and so integrate the equation

$$x_2x_3dx_1 + (x_3x_1 + x_3x_4 + x_2x_3)\,dx_2 + (x_1x_2 + x_2x_3 + x_2x_4)\,dx_3 + x_2x_3dx_4 = 0.$$

(Tanner.)

70. In the case of an even number of original variables the following simplification, due to Jacobi*, is worthy of notice. Each of the elements in the reduced form is an integral of a

* See the memoir "Ueber die Reduction der Integration der partiellen Differentialgleichungen erster Ordnung zwischen irgend einer Zahl Variabeln auf die Integration eines einzigen Systemes gewöhnlicher Differentialgleichungen." *Crelle*, t. xvii. (1837), pp. 97—162, especially § 12, pp. 156—162; or in the Collected Works, vol. iv., pp. 57—127, especially pp. 120—127.

subsidiary system such as (10); and so for each of the stages there is the implication that a subsidiary system is to be formed. The proper choice of the integrals of the successive subsidiary systems can be made effective to avoid the successive transformations of the differential expression: the essential idea being the introduction of what are called "initial values" of the variables suggested first by Cauchy and afterwards by Hamilton, to whom Jacobi attributes it*.

Since the number of variables in the original Pfaffian expression is even, the subsidiary system of § 60 has $2n-1$ integrals: let these be

$$u_r(x_1, \ldots\ldots, x_{2n-1}, x_{2n}) = \text{constant} = a_r,$$

for $r = 1, 2, \ldots, 2n-1$. The right-hand side being constant, its value will not be affected, if we assign to any variable a particular value, say c_{2n} to x_{2n}. This assignation is a limitation on the variations; since there are $2n-1$ equations in $2n$ variables, one of which is (temporarily) determinate, the others will also be determinate and so we take c_i (for $i = 1, 2, \ldots, 2n-1$) as the simultaneous values of x_i. Hence we have first

$$u_r(c_1, c_2, \ldots\ldots, c_{2n-1}, c_{2n}) = a_r$$

for $r = 1, \ldots, 2n-1$; so that $c_1, \ldots, c_{2n-1}$ are functions of $a_1, \ldots, a_{2n-1}$, and so are constants which, when the values of $a_1, \ldots, a_{2n-1}$ in the forms $u(x_1, \ldots, x_{2n})$ are substituted, lead to functions of the variables and constitute integral equations. These $2n-1$ integral equations of the subsidiary system may be taken in the form

$$c_i = v_i(x_1, \ldots, x_{2n}) = v_i$$

for $i = 1, \ldots, 2n-1$, where v_i reduces to x_i when $x_{2n} = c_{2n}$†. And secondly, we have the $2n-1$ equations

$$u_r(x_1, \ldots, x_{2n-1}, x_{2n}) = a_r = u_r(c_1, \ldots, c_{2n-1}, c_{2n}),$$

so that solving for $x_1, \ldots, x_{2n-1}$ we have

$$\begin{aligned} x_s &= f_s(c_1, \ldots, c_{2n-1}, c_{2n}, x_{2n}) \\ &= f_s(v_1, \ldots, v_{2n-1}, c_{2n}, x_{2n}). \end{aligned}$$

When these values are substituted in the expression Ω, we find a

* See note to § 109, post.

† These are 'principal integrals' of the system.

result of the form

$$\Omega = B dx_{2n} + \lambda \sum_{i=1}^{2n-1} C_i \, dv_i .$$

In this form we have $B = 0$, because the quantities v_i are integrals of the subsidiary system chosen so as to lead to the evanescence of the differential element dx_{2n}; and the variable x_{2n} occurs explicitly only, if at all, in λ. Hence

$$\sum_{i=1}^{2n} X_i dx_i = \Omega = \lambda \sum_{i=1}^{2n-1} C_i \, dv_i .$$

Now the right-hand side is, except possibly in the factor λ, independent of x_{2n}; and therefore, except in that factor, will not be altered by the assignation to x_{2n} of any special value, say c_{2n}. The consequent values of $v_1, \ldots, v_{2n-1}$ are the degenerate forms of those variables for the special value of x_{2n}, being $x_1, \ldots, x_{2n-1}$ respectively; and therefore

$$\begin{aligned} \sum_{i=1}^{2n-1} [X_i]_{x_{2n}=c_{2n}} dx_i = [\Omega]_{x_{2n}=c_{2n}} &= [\lambda]_{x_{2n}=c_{2n}} \sum_{i=1}^{2n-1} [C_i]_{v=x} \, dx_i \\ &= \lambda' \sum_{i=1}^{2n-1} [C_i]_{v=x} \, dx_i \end{aligned}$$

say. This being an identity, the coefficients of the differential variations are equal to one another on the two sides of this equation, so that

$$\lambda' [C_i]_{v=x} = [X_i]_{x_{2n}=c_{2n}},$$

a result which gives the *form* of C_i, and is equivalent to the result which would be obtained by substitution. Since it is valid for all variables, we change the variables x_i into v_i; and then $[C_i]_{v=x}$ becomes C_i. Let λ' become λ'', which thus is the value of λ when for $x_1, \ldots, x_{2n-1}, x_{2n}$ we substitute $v_1, \ldots, v_{2n-1}, c_{2n}$; and let V_i denote the consequent form of $[X_i]_{x_{2n}=c_{2n}}$, so that *when we make these same substitutions in X_i we denote the result by V_i*. Thus

$$\lambda'' C_i = V_i,$$

and therefore

$$\begin{aligned} \Omega &= \lambda \sum_{i=1}^{2n-1} C_i \, dv_i \\ &= \frac{\lambda}{\lambda''} \sum_{i=1}^{2n-1} V_i \, dv_i , \end{aligned}$$

where V_i is derived from X_i merely by the substitution of $v_1, \ldots, v_{2n-1}, c_{2n}$ for $x_1, \ldots, x_{2n-1}, x_{2n}$ respectively.

It thus appears that when the integrals of the subsidiary Pfaffian system are taken in the form of "principal" integrals, the transformed expression of Ω (save as to a factor which is negligible from the point of view of the integral equivalent of Ω) is derivable from the original form of Ω merely by a literal change.

In obtaining the integral equivalent, we take one of these new variables v as an integral (§§ 68, 69), say

$$v_{2n-1} = a_{2n-1};$$

after which the differential equation becomes

$$\sum_{i=1}^{2n-2} V'_i \, dv_i = 0,$$

where V'_i *is the value of* V_i with a_{2n-1} substituted in it for v_{2n-1}, that is, is the value of X_i *with* $v_1, \ldots, v_{2n-2}, a_{2n-1}, c_{2n}$ *substituted in it for* $x_1, \ldots, x_{2n-2}, x_{2n-1}, x_{2n}$ *respectively,* a_{2n-1} and c_{2n} being reckoned constants.

The characteristic form of the new (reduced) equation is the same as that of the original equation; and the new coefficients are derived by merely literal changes from the original coefficients, so that much of the analysis leading to the equations, subsidiary to transformation, will after the same literal changes be useful for the succeeding system of subsidiary equations.

Ex. Shew that, if $u_r = a_r$ $(r = 1, 2, \ldots, n)$ be n integral equations satisfying

$$X_1 dx_1 + X_2 dx_2 + \ldots\ldots + X_{2n} dx_{2n} = 0,$$

and if n new integral equations be formed, first by solving for $x_1, x_2, \ldots, x_n$ in terms of $x_{n+1}, \ldots, x_{2n}, a_1, \ldots, a_n$, say

$$x_r = f_r(x_{n+1}, \ldots\ldots, x_{2n}, a_1, \ldots\ldots, a_n),$$

and next by constructing

$$X_1 \frac{\partial f_1}{\partial a_r} + X_2 \frac{\partial f_2}{\partial a_r} + \ldots\ldots + X_n \frac{\partial f_n}{\partial a_r} + \lambda b_r = 0, \quad (r = 1, \ldots\ldots, n),$$

then the $2n-1$ equations, resulting from the elimination of λ among the last n equations and the original n equations, are equations containing $2n-1$ arbitrary constants, viz. $a_1, \ldots, a_n, b_1/b_n, \ldots, b_{n-1}/b_n$ and are the complete integral of the Pfaffian system subsidiary to the even reduction of the differential equation.

(Jacobi.)

CHAPTER V.

GRASSMANN'S METHOD*.

71. THE variables $x_1, \ldots, x_m$ of the Pfaffian equation

$$X_1dx_1 + X_2dx_2 + \ldots\ldots + X_mdx_m = 0$$

are independent, and the coefficients X are functions of the variables x. To apply the processes of the Ausdehnungslehre to this equation, we consider a region (Hauptgebiet) of m simple units say $e_1, \ldots, e_m$; and from them and the variables we construct an extensive magnitude x in the form

$$x = x_1e_1 + x_2e_2 + \ldots\ldots + x_me_m,$$

a magnitude of ordinal (Stufenzahl) unity. We take the comple-

* Grassmann's method of dealing with Pfaff's equation is to be found in §§ 500—527 of the 1862 edition of the Ausdehnungslehre.

The present chapter has been included only after very considerable hesitation, which arose not from any doubt about the substantial character of the contribution to the theory of the Pfaffian equation effected by Grassmann, but from a feeling that it is not possible to give an intelligible account of his method without either assuming, what is probably not common at present, some adequate knowledge of the analytical method of the Ausdehnungslehre or giving some adequate explanation of the method. The latter would require more space than could fairly be granted in the present treatise; and so it happens that what is to be found in the present chapter is little more than a translation of the above-quoted sections. In order to be understood, it must be read on the basis of an acquaintance with Grassmann's analytical method.

The reason for the inclusion of the chapter, in spite of the foregoing very serious objection, is the necessity of rendering justice to Grassmann in an acknowledgement of the definite results and, in some details, of the definite novelties (§ 48) obtained by his analysis. So far as concerns the present question, his results are always remarkably concise in form, as may be seen by a comparison with the expression of the same results in other methods; their difficulty lies, partly in their interpretation, partly in the proofs which are not always clearly explained.

mentaries of the units e, denoting them by $E_1, \ldots, E_m$; and construct another extensive magnitude X in the form

$$X = X_1E_1 + X_2E_2 + \ldots\ldots + X_mE_m,$$

a magnitude of ordinal $m-1$. Then we have

$$\begin{aligned} Xdx &= (X_1E_1 + \ldots\ldots + X_mE_m)(e_1dx_1 + \ldots\ldots + e_mdx_m) \\ &= (-1)^{m-1}(X_1dx_1 + X_2dx_2 + \ldots\ldots + X_mdx_m) \end{aligned}$$

a numerical quantity (Zahlgrösse). Hence *the original differential equation can be replaced by*

$$Xdx = 0,$$

in which x is an extensive magnitude and X can be expressed as a function of x, but Xdx is a numerical quantity.

This result includes the partial differential equation of the first order as a special case. The result can be extended to partial differential equations of order higher than the first or, what is the same thing, to systems of simultaneous Pfaffians; and Grassmann shews that a whole system can be replaced by a single equation

$$Xdx = 0,$$

where x is an extensive magnitude, X is a function of x and Xdx is also an extensive magnitude.

72. Other investigations (§ 69) connected with the Pfaffian equation shew that its integral equivalent is composed of a set of simultaneous equations, assumed to be the smallest number, in virtue of which the differential equation in the ordinary form can be satisfied. Let then

$$u_1 = c_1,\ u_2 = c_2,\ \ldots\ldots,\ u_n = c_n$$

be such a system of equations in numerical quantities, in virtue of which the differential equation is satisfied: then there must exist quantities U such that

$$Xdx = U_1du_1 + U_2du_2 + \ldots\ldots + U_ndu_n.$$

For since the foregoing system of integral equations is the smallest number, in virtue of which

$$Xdx = 0$$

is satisfied, it follows that

$$du_1 = 0,\ du_2 = 0,\ \ldots\ldots,\ du_n = 0$$

is the smallest number of exact equations, in virtue of which the

differential equation subsists. Since all the equations are linear in the differential elements of the variables, the original equation can be only a linear combination of the n exact equations, so that we must have quantities U such that

$$Xdx = U_1du_1 + U_2du_2 + \ldots\ldots + U_ndu_n.$$

All the elements du must occur: otherwise the equation would be satisfied by the vanishing of those which do occur, that is, by an integral system containing fewer members*.

73. We have, by the definition of a differential coefficient with respect to an extensive magnitude,

$$\frac{du}{dx}dx = du,$$

and therefore

$$\sum_{i=1}^{n} U_idu_i = \sum_{i=1}^{n} U_i\frac{du_i}{dx}dx,$$

so that we may take

$$X = \sum_{i=1}^{n} U_i\frac{du_i}{dx},$$

and X is an expression with a single gap (Lücke). Hence

$$\frac{dX}{dx} = \sum_{i=1}^{n}\frac{dU_i}{dx}\frac{du_i}{dx} + \sum_{i=1}^{n} U_i\frac{d^2u_i}{dx^2},$$

an expression with a double gap; and some of the parts of it, viz. each quantity $\frac{d^2u}{dx^2}$, have the two gaps interchangeable.

When we proceed to form the interrupted (lückenhaltig) product represented by

$$\left[X\left(\frac{dX}{dx}\right)^n\right],$$

we may omit from the $\frac{dX}{dx}$ those parts which have two interchangeable gaps, because they lead to vanishing quantities. Hence

$$\left[X\left(\frac{dX}{dx}\right)^n\right] = \left[\Sigma\, U_i\frac{du_i}{dx}\left(\Sigma\frac{dU_j}{dx}\frac{du_j}{dx}\right)^n\right]$$
$$= \Sigma\left[U_i\frac{du_i}{dx}\frac{dU_j}{dx}\frac{du_j}{dx}\frac{dU_k}{dx}\frac{du_k}{dx}\ldots\right],$$

* For the more general form of the integral system in virtue of which ΣUdu vanishes, see § 142, note.

where the number of different indices $i, j, k, \ldots$ is $n+1$, and the bracket implies the usual law of interrupted multiplication. Now since there are only n quantities u and there are $n+1$ indices, it follows that two of the quantities $\dfrac{du}{dx}$ in any of the combinatory products are the same and therefore each such product is zero. Hence every term of the sum vanishes; and *therefore*

$$\left[X\left(\frac{dX}{dx}\right)^n\right]=0,$$

if Xdx be expressible as in the last paragraph.

If the same process be applied, so as to form

$$\left[X\left(\frac{dX}{dx}\right)^{n-1}\right],$$

we cannot in general assert that it will vanish: nor, if it be applied so as to form

$$\left[\left(\frac{dX}{dx}\right)^n\right],$$

can we in general assert that it will vanish*; but, *if it be applied so as to form*

$$\left[\left(\frac{dX}{dx}\right)^{n+1}\right],$$

it is easy to see that the expression must vanish.

74. The two equations, symbolical in the interrupted products, just obtained, viz.

$$\left[X\left(\frac{dX}{dx}\right)^n\right]=0,\quad \left[\left(\frac{dX}{dx}\right)^{n+1}\right]=0,$$

can be replaced by numerical equations.

Let us first consider the former. Since X has a single gap and $\dfrac{dX}{dx}$ a double gap, we have

$$\left[X\left(\frac{dX}{dx}\right)^n e_\alpha e_\beta \ldots e_\iota e_\theta\right]=0,$$

where $\alpha, \beta, \ldots, \iota, \theta$ are any $2n+1$ of the integers in the series $1, \ldots, m$.

* The expression will vanish, if the quantities U and u be not independent of one another; see references in note to § 63.

First, if $m \overline{\gtrless} 2n$, then at least some one of the quantities e can be expressed linearly in terms of the remaining ones by means of a relation, the coefficients of which are numerical; hence the product is necessarily zero, and so there is no condition to be inferred.

Secondly, let $m = 2n + 1$; then there is only one equation

$$\left[X\left(\frac{dX}{dx}\right)^n e_1e_2 \ldots e_{2n+1}\right] = 0.$$

Now, except as to a power $(-1)^{m-1}$, we have

$$Xe_\alpha = X_\alpha,$$

and therefore

$$\frac{dX}{dx} e_\beta e_\gamma = \frac{d}{dx} Xe_\beta e_\gamma = \frac{d}{dx} X_\beta e_\gamma = \frac{\partial X_\beta}{\partial x_\gamma}.$$

The foregoing expression, being an interrupted product, is thus equal to

$$\Sigma\left(\pm X_\alpha \frac{\partial X_\beta}{\partial x_\gamma} \cdots\cdots \frac{\partial X_\iota}{\partial x_\theta}\right) = 0$$

except as to a dropped numerical factor equal to $(2n+1)!$, the summation extending to all derangements of the indices from the sequence 1, 2, ..., $2n+1$, and a positive or a negative sign being associated with a term according as it arises from an even or from an odd derangement of the sequence. If the derangements be taken in pairs, such that in each pair the derangements are one odd and the other even and are the same except for the two deranged indices, then in such a case we can combine in pairs. For instance, if we have one term

$$+ X_\alpha \frac{\partial X_\beta}{\partial x_\gamma} \frac{\partial X_\delta}{\partial x_\epsilon} \cdots\cdots \frac{\partial X_\iota}{\partial x_\theta},$$

we shall have another term

$$- X_\alpha \frac{\partial X_\gamma}{\partial x_\beta} \frac{\partial X_\delta}{\partial x_\epsilon} \cdots\cdots \frac{\partial X_\iota}{\partial x_\theta},$$

which two added together give

$$+ X_\alpha a_{\beta\gamma} \frac{\partial X_\delta}{\partial x_\epsilon} \cdots\cdots \frac{\partial X_\iota}{\partial x_\theta}.$$

Proceeding in this way we shall have

$$\Sigma \pm (X_\alpha a_{\beta\gamma} a_{\delta\epsilon} \cdots\cdots a_{\iota\theta}) = 0$$

which, with the now limited derangements, coincides with the Jacobian condition (§ 65).

Thirdly, if $m > 2n+1$, we have the equation

$$\left[X\left(\frac{dX}{dx}\right)^n e_\alpha e_\beta \ldots\ldots e_\iota e_\theta\right] = 0$$

subsisting for every selection of $2n+1$ integers $\alpha, \beta, \ldots, \iota, \theta$ from the set $1, \ldots, m$; and every such symbolical equation leads to a numerical equation

$$\Sigma \pm (X_\alpha a_{\beta\gamma} \ldots\ldots a_{\iota\theta}) = 0.$$

Thus the number of numerical equations of this form is the same as the number of selections of $2n+1$ integers from the system $1, \ldots, m$; but they are not all independent, and it is sufficient to retain only those which have some specified index, say 1, common and which therefore are in number equal to the number of selections of $2n$ integers from the series $2, \ldots, m$.

The second equation of § 73

$$\left[\left(\frac{dX}{dx}\right)^{n+1}\right] = 0$$

may be treated in the same way: it leads to

$$\left[\left(\frac{dX}{dx}\right)^{n+1} e_1 e_2 \ldots\ldots e_{2n+2}\right] = 0$$

for every selection of $2n+2$ integers from the series $1, \ldots, m$.

If m be less than $2n+2$, the product is necessarily zero.

If m be equal to $2n+2$, then as before we are led to the numerical condition

$$\Sigma \pm [a_{12} \ldots\ldots a_{2n+1,\, 2n+2}] = 0$$

with the same laws of summation.

If m be greater than $2n+2$, then there is for every selection as above indicated a numerical equation of the form given for the case $m = 2n+2$.

These conditions are not independent of the former, of which important property Grassmann gives two proofs. It is not necessary to repeat them here and it may be sufficient to point out that his first proof is, after the translation of his analysis into numerical forms, equivalent to a proof of the identity

$$X_1\left[\left(\frac{dX}{dx}\right)^{n+1} e_1 \ldots e_{2n+2}\right] = \sum_{i=2}^{2n+2} a_{1i}\left[X\left(\frac{dX}{dx}\right)^n e_{i+1}e_{i+2}\ldots e_{i-1}\right],$$

where on the right-hand side 1 is omitted from the series $i+1,\ i+2,\ \ldots,\ i-1$: the establishment of the identity is a justification of the statement.

Ex. If the Pfaffian equation

$$\sum_{i=1}^{m} X_i dx_i = 0$$

can be satisfied by a single equation $u=c$, so that it takes the form

$$\sum_{i=1}^{m} X_i dx_i = U du = 0,$$

then the conditions for this are the aggregate contained in the symbolical equation

$$\left[X\frac{dX}{dx}\right]=0,$$

which, when numerically interpreted as above, are the aggregate

$$X_\alpha a_{\beta\gamma} + X_\beta a_{\gamma\alpha} + X_\gamma a_{\alpha\beta} = 0$$

for every three indices $\alpha,\ \beta,\ \gamma$.

75. When the conditions necessary that the equation

$$X dx = 0$$

should have an integral system of n equations have been obtained, the next process is the transformation of the numerical quantity $X dx$; and, as in other methods which treat the Pfaffian equation, the transformation is gradual.

The extensive variable x, constructed from m units, is to be expressed as a function of another extensive variable a, constructed from only $m-1$ units, and of a numerical variable t; and the transformation is to be determined so that, when substitution takes place for x in the equation, the expression $X dx$ is to be independent of dt and, save possibly as to a numerical factor N, independent of t. If then we take

$$dx = d_a x + d_t x,$$

we are to have

$$\left.\begin{aligned} X d_t x &= 0 \\ d_t \frac{1}{N} X d_a x &= 0 \end{aligned}\right\}.$$

Writing

$$\lambda = \frac{d_t N}{N},$$

the second equation gives

$$\lambda X d_a x = d_t (X d_a x) = d_t X d_a x + X d_t d_a x.$$

Also by the first equation we have

$$0 = d_a (X d_t x) = d_a X d_t x + X d_a d_t x,$$

and t is numerical so that

$$d_a d_t x = d_t d_a x;$$

hence by subtraction we have

$$\begin{aligned} \lambda X d_a x &= d_t X d_a x - d_a X d_t x \\ &= \frac{dX}{dx} d_t x \, d_a x - \frac{dX}{dx} d_a x \, d_t x, \end{aligned}$$

or

$$-\lambda X d_a x = \left[\frac{dX}{dx} d_a x \, d_t x\right],$$

the right-hand side now being an interrupted product, in which $\dfrac{dX}{dx}$ has a double gap.

Then we have the proposition:—

If c denote any quantity of the same species as x and $d_a x$ and if, for every such quantity c, the equation

$$-\lambda X c = \left[\frac{dX}{dx} c d_t x\right]$$

be satisfied, then

$$X d_t x = 0 \text{ and } d_t \frac{1}{N} X d_a x = 0,$$

where $\dfrac{d_t N}{N} = \lambda$ *which is supposed to be different from zero.*

First, since the equation

$$-\lambda X c = \left[\frac{dX}{dx} c d_t x\right]$$

is satisfied for every quantity c of the species indicated and since $d_t x$ is such a quantity, we have

$$\begin{aligned} -\lambda X d_t x &= \left[\frac{dX}{dx} d_t x \, d_t x\right] \\ &= 0. \end{aligned}$$

Also λ is numerical and different from zero: hence

$$Xd_tx = 0,$$

which proves the first of the equations. From it we have

$$\begin{aligned} 0 &= d_aXd_tx + Xd_ad_tx \\ &= \frac{dX}{dx} d_axd_tx + Xd_ad_tx. \end{aligned}$$

Again in the equation, which is satisfied for the quantities c, substitute for c a quantity d_ax which is of the implied species; thus we obtain

$$\begin{aligned} -\lambda Xd_ax &= \left[\frac{dX}{dx} d_axd_tx\right] \\ &= \frac{dX}{dx} d_axd_tx - \frac{dX}{dx} d_txd_ax, \end{aligned}$$

which, when subtracted from the result just obtained, leads to

$$\begin{aligned} \lambda Xd_ax &= Xd_ad_tx + \frac{dX}{dx} d_txd_ax \\ &= Xd_ad_tx + d_tXd_ax \\ &= Xd_td_ax + d_tXd_ax \\ &= d_t(Xd_ax) \end{aligned}$$

and hence

$$d_t\left(\frac{1}{N} Xd_ax\right) = 0,$$

proving the second of the equations. In fact the whole proof is little more than a reversal of the earlier investigation.

Since

$$\begin{aligned} Xdx &= Xd_ax + Xd_tx \\ &= Xd_ax, \end{aligned}$$

we have

$$Xd_ax = 0,$$

and therefore, as N is numerical, also

$$\frac{1}{N} Xd_ax = 0,$$

an equation which involves neither t nor dt and is therefore the transformed form of $Xdx = 0$.

It thus appears that the satisfaction of the characteristic equation

$$-\lambda X c = \left[\frac{dX}{dx}\, c d_t x\right]$$

is sufficient to lead to the required transformation.

76. We now proceed to find a value of $d_t x$, such that the characteristic equation

$$-\lambda X c = \left[\frac{dX}{dx}\, c d_t x\right]$$

is satisfied for all quantities c of the same species as x, that is, is linearly constructible from the units which enter into x. The method adopted for this purpose is to find $d_t x$ for one particular quantity c of the appropriate species and to shew that the inferred value of $d_t x$ allows the equation to be satisfied for all such quantities.

In the first place we assume that the equations

$$\left[X\left(\frac{dX}{dx}\right)^{n-1}\right] = 0, \qquad \left[\left(\frac{dX}{dx}\right)^{n}\right] = 0$$

are not both satisfied, for these are the conditions that the original Pfaffian equation $Xdx = 0$ should be satisfied by $n-1$ integrals; hence $m > 2n - 1$. Also the set of equations

$$\left[\left(\frac{dX}{dx}\right)^{n}\right] = 0$$

would, if satisfied, be a consequence of the set

$$\left[X\left(\frac{dX}{dx}\right)^{n-1}\right] = 0;$$

and therefore it will be assumed that

$$\left[X\left(\frac{dX}{dx}\right)^{n-1}\right] \gtrless 0,$$

as is justifiable.

77. First, let $m = 2n$; and in the meanwhile assume that $\left[\left(\frac{dX}{dx}\right)^{n}\right]$ does not vanish.

As in § 71, let E_r denote the complementary of e_r so chosen that $[e_r E_r] = 1$; evidently E_r is the product of $2n - 1$ units, and may be taken $(-1)^{r-1} e_{r+1} \ldots e_{2n} e_1 \ldots e_{r-1}$.

Since $\dfrac{dX}{dx}$ is an expression with a double gap, the quantity

$$\left[\left(\frac{dX}{dx}\right)^{n-1} E_r\right]$$

is linear in the units e and is therefore of the same species as x; hence it may be substituted in the equation, which thus becomes

$$-\lambda X \left[\left(\frac{dX}{dx}\right)^{n-1} E_r\right] = \left[\frac{dX}{dx}\left[\left(\frac{dX}{dx}\right)^{n-1} E_r\right] d_t x\right].$$

Since X is an expression with a single gap and $\left[\left(\dfrac{dX}{dx}\right)^{n-1} E_r\right]$ a quantity which is linear in the units, it follows that

$$X\left[\left(\frac{dX}{dx}\right)^{n-1} E_r\right]$$

is a numerical magnitude, and that it can be expressed in the form

$$\left[X\left(\frac{dX}{dx}\right)^{n-1} E_r\right].$$

Similarly the right-hand side is a numerical quantity which can be expressed in the form

$$\left[\left(\frac{dX}{dx}\right)^{n} E_r d_t x\right];$$

and therefore the equation is

$$-\lambda\left[X\left(\frac{dX}{dx}\right)^{n-1} E_r\right] = \left[\left(\frac{dX}{dx}\right)^{n} E_r d_t x\right]$$

$$= \left[\left(\frac{dX}{dx}\right)^{n} E_r \sum_{i=1}^{2n} e_i d_t x_i\right].$$

Now, when i is different from r, we have $[E_r e_i] = 0$; and also

$$[E_r e_r] = (-1)^{2n-1}[e_r E_r] = -1;$$

thus we have

$$\lambda\left[X\left(\frac{dX}{dx}\right)^{n-1} E_r\right] = \left[\left(\frac{dX}{dx}\right)^{n}\right] d_t x_r.$$

The coefficient of λ on the left-hand side being numerical and involving $2n-1$ indices, it will (when interpreted numerically) be an algebraical sum divided by $(2n-1)!$. Similarly the coefficient of $d_t x_r$ on the right-hand side will (when interpreted numerically) be an algebraical sum divided by $2n!$. Hence taking

$$\mu = \frac{2n!\ \lambda}{\left[\left(\frac{dX}{dx}\right)^n\right]} \frac{1}{(2n-1)!} = \frac{2\,n\lambda}{\left[\left(\frac{dX}{dx}\right)^n\right]},$$

we have

$$d_t x_r = \frac{\mu}{2n}\left[X\left(\frac{dX}{dx}\right)^{n-1} E_r\right] \;\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and therefore

$$\begin{aligned} d_t x &= \sum_{i=1}^{2n} e_i d_t x_i \\ &= \frac{\mu}{2n}\sum_{i=1}^{n} e_i\left[X\left(\frac{dX}{dx}\right)^{n-1} E_r\right] \\ &= \mu\left[X\left(\frac{dX}{dx}\right)^{n-1}\right] \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{I}), \end{aligned}$$

which gives a value for $d_t x$.

78. It is now necessary to prove that this value of $d_t x$ will permit the equation in c to be satisfied, whatever quantity of the appropriate species be substituted for c.

Since X has a single gap and $\frac{dX}{dx}$ has a double gap, the interrupted product $\left[X\left(\frac{dX}{dx}\right)^{n-1}\right]$ has $2n-1$ gaps, while there are $2n$ units e; hence it is a numerical quantity, when its gaps are occupied by any set of all the units save one. Since there is thus one unit left, we may associate with $\left[X\left(\frac{dX}{dx}\right)^{n-1}\right]$ a factor unity and assign a gap to this factor without altering the signification; and so

$$\begin{aligned} \left[X\left(\frac{dX}{dx}\right)^{n-1}\right] &= \left[\iota X\left(\frac{dX}{dx}\right)^{n-1}\right] \\ &= -\left[X\iota\left(\frac{dX}{dx}\right)^{n-1}\right] \end{aligned}$$

$$= -\frac{1}{2n}\sum_{\alpha=1}^{2n}\left[Xe_\alpha[l\left(\frac{dX}{dx}\right)^{n-1}E_\alpha]\right]$$
$$= -\frac{1}{2n}\sum_{\alpha=1}^{2n}Xe_\alpha\left[l\left(\frac{dX}{dx}\right)^{n-1}E_\alpha\right],$$

since Xe_α is numerical. Hence, substituting this in the value (I) of d_tx, we have

$$\left[\frac{dX}{dx}e_rd_tx\right] = -\frac{\mu}{2n}\sum_{\alpha=1}^{2n}Xe_\alpha\left[\frac{dX}{dx}e_r[l\left(\frac{dX}{dx}\right)^{n-1}E_\alpha]\right]$$
$$= -\frac{\mu}{2n}\sum_{\alpha=1}^{2n}Xe_\alpha\left[\left(\frac{dX}{dx}\right)^n e_rE_\alpha\right].$$

Now when α is different from r, we have $[e_rE_\alpha]=0$, and when α is r, we have $[e_rE_r]=1$; hence

$$\left[\frac{dX}{dx}e_rd_tx\right] = -\frac{\mu}{2n}Xe_r\left[\left(\frac{dX}{dx}\right)^n\right]$$
$$= -\lambda Xe_r$$

on substituting the value of μ. This holds for every index r; and since every quantity c of the assigned species is of the form

$$c = \sum_{r=1}^{2n} e_ry_r,$$

where the y's are numerical, we have

$$\left[\frac{dX}{dx}cd_tx\right] = -\lambda Xc,$$

or the value of d_tx is consistent with the satisfaction of the equation, and is therefore sufficient to lead to the required transformation of Xdx.

The equations (1), being numerical equations, can be put into the following form. We have

$$(-1)^{r-1}E_r = e_1e_2\ldots\ldots e_{r-1}e_{r+1}\ldots\ldots e_{2n};$$

and therefore as in § 74

$$\left[X\left(\frac{dX}{dx}\right)^{n-1}E_r\right] = \frac{(-1)^{r-1}}{(2n-1)!}\Sigma\pm(X_1a_{23}\ldots\ldots a_{2n-1,\,2n})$$

with the same derangements of the indices occurring in the summation, the index r being omitted. For example, let $n=3$; then

$$\left[X\left(\frac{dX}{dx}\right)^2E_1\right] = \frac{1}{120}\{X_2(a_{34}a_{56}-a_{35}a_{46}+a_{36}a_{45}) - X_3(a_{24}a_{56}-a_{25}a_{46}+a_{26}a_{45})$$
$$+X_4(a_{23}a_{56}-a_{25}a_{36}+a_{26}a_{35}) - X_5(a_{23}a_{46}-a_{24}a_{36}+a_{26}a_{34})$$
$$+X_6(a_{23}a_{45}-a_{24}a_{35}+a_{25}a_{34})\},$$

$$\left[X\left(\frac{dX}{dx}\right)^2 E_2\right] = \frac{-1}{120}\{X_1(a_{34}a_{56} - a_{35}a_{46} + a_{36}a_{45}) - X_3(a_{14}a_{56} - a_{15}a_{46} + a_{16}a_{45})$$
$$+ X_4(a_{13}a_{56} - a_{15}a_{36} + a_{16}a_{35}) - X_5(a_{13}a_{46} - a_{14}a_{36} + a_{16}a_{34})$$
$$+ X_6(a_{13}a_{45} - a_{14}a_{35} + a_{15}a_{34})\}$$

and so on. And the subsidiary equations (1) take the form

$$\frac{dx_1}{\left[X\left(\frac{dX}{dx}\right)^{n-1} E_1\right]} = \frac{dx_2}{\left[X\left(\frac{dX}{dx}\right)^{n-1} E_2\right]} = \frac{dx_3}{\left[X\left(\frac{dX}{dx}\right)^{n-1} E_3\right]} = \dots\dots = \frac{\mu}{2n}$$

which, after the above expansions, are easily seen to agree with the first subsidiary Pfaffian system as obtained by ordinary analysis.

79. It was assumed in what precedes that $\left[\left(\frac{dX}{dx}\right)^n\right]$ does not vanish. Taking now the alternative supposition that

$$\left[\left(\frac{dX}{dx}\right)^n\right] = 0,$$

we have any of the subsidiary equations of § 77 in the earlier form

$$\lambda\left[X\left(\frac{dX}{dx}\right)^{n-1} E_r\right] = \left[\left(\frac{dX}{dx}\right)^n\right] d_t x_r$$
$$= 0,$$

and the coefficient of λ does not vanish; *hence λ itself vanishes* and so N is independent of t. Therefore, if the transformation be possible, it is to be expected to be such as to leave the new form of Xdx free, not merely from dt, but also from t.

Now, just as in the corresponding case in the reduction of §§ 61—63 with an even number of original variables, the subsidiary equations (and consequently the equation which gives $d_t x$) are valid, though obtained on a supposition which no longer holds: that is, we have for $m = 2n$,

$$d_t x = \mu\left[X\left(\frac{dX}{dx}\right)^{n-1}\right]$$

even when the Pfaffian $\left[\left(\frac{dX}{dx}\right)^n\right]$ vanishes (§ 74). The proof is as follows.

We have, for this value of $d_t x$,

$$Xd_t x = \mu X\left[X\left(\frac{dX}{dx}\right)^{n-1}\right]$$
$$= \mu\left[X^2\left(\frac{dX}{dx}\right)^{n-1}\right].$$

When we form this interrupted product as in § 73, we shall have in it $n+1$ factors involving only n quantities $\frac{du}{dx}$, so that every term will contain some quantity $\frac{du}{dx}$ at least twice and must therefore vanish; hence the sum will vanish, and we have

$$Xd_tx = 0,$$

the first of the essential equations.

Next, we have from the value of d_tx, as in § 78,

$$\left[\frac{dX}{dx} e_r d_t x\right] = -\frac{\mu}{2n} X e_r \left[\left(\frac{dX}{dx}\right)^n\right] = 0,$$

and therefore for any quantity c of the same species as x we have

$$\left[\frac{dX}{dx} c d_t x\right] = 0.$$

Now $d_a x$ is of this kind and therefore

$$\left[\frac{dX}{dx} d_a x d_t x\right] = 0,$$

that is,

$$\frac{dX}{dx} d_a x d_t x - \frac{dX}{dx} d_t x d_a x = 0,$$

and therefore

$$d_a X \,.\, d_t x - d_t X \,.\, d_a x = 0\,;$$

whence by the addition and subtraction in the left-hand side of the equal quantities $d_a d_t x$ and $d_t d_a x$ we have

$$d_a (X d_t x) - d_t (X d_a x) = 0.$$

But we have proved that

$$Xd_t x = 0\,;$$

hence

$$d_t (X d_a x) = 0,$$

so that $Xd_a x$ is independent of t. Moreover

$$\begin{aligned} Xdx &= Xd_a x + Xd_t x \\ &= Xd_a x, \end{aligned}$$

so that *the transformation determined by the equation*

$$d_t x = \mu \left[X \left(\frac{dX}{dx}\right)^{n-1}\right]$$

makes the new form of Xdx independent not merely of dt, but also, in the case when $\left[\left(\frac{dX}{dx}\right)^n\right] = 0$, *of* t.

80. The equation

$$d_t x = \mu \left[X\left(\frac{dX}{dx}\right)^{n-1}\right]$$

determines the derivatives $d_t x_1, \ldots, d_t x_{2n}$, and μ; but as it is equivalent only to $2n$ equations, we may consider one of the quantities left undetermined and therefore assumable at will. We take $t = x_{2n}$, the assumption adopted in other methods applied to the Pfaffian equation; which, substituted in equation (1) of § 77 subsisting for $r = 2n$, gives

$$\mu = \frac{2n\,dx_{2n}}{\left[X\left(\frac{dX}{dx}\right)^{n-1} E_{2n}\right]}.$$

Let

$$y = x_1 e_1 + x_2 e_2 + \ldots + x_{2n-1} e_{2n-1},$$

so that

$$y = x - x_{2n} e_{2n}.$$

Then

$$d_{x_{2n}} y = d_{x_{2n}} x - e_{2n}\, dx_{2n}$$

$$= \frac{2n\,dx_{2n}}{\left[X\left(\frac{dX}{dx}\right)^{n-1} E_{2n}\right]} \left[X\left(\frac{dX}{dx}\right)^{n-1}\right] - e_{2n}\, dx_{2n}$$

or

$$\frac{dy}{dx_{2n}} = \frac{2n}{\left[X\left(\frac{dX}{dx}\right)^{n-1} E_{2n}\right]} \left[X\left(\frac{dX}{dx}\right)^{n-1}\right] - e_{2n}.$$

When on the right-hand side we substitute $y + x_{2n} e_{2n}$ for x, it comes to be a function of y and x_{2n}.

This equation on integration gives the result first in the form

$$y = a + x_{2n} \phi,$$

where ϕ is a function of y, x_{2n} and a, and where a is the value of y when x_{2n} is zero, that is, it is the value of x when x_{2n} is zero;

and so it involves only $2n-1$ of the units: and then we have a in the form

$$\begin{aligned} a &= \text{function of } y \text{ and } x_{2n} \\ &= \text{function of } x \text{ and } x_{2n} \end{aligned}$$

after substitution for y*.

81. Let the numerical quantity N, in which x_{2n} occurred if it occurred anywhere, be denoted by $N(x)$ when expressed as a function of x. We have

$$x = a + x_{2n}(\phi + e_{2n}),$$

and therefore

$$d_a x = da + x_{2n} d_a \phi.$$

Now the transformed equation, which is

$$\frac{1}{N} X d_a x = 0,$$

thus comes to be

$$\frac{1}{N(x)} X(x)(da + x_{2n} d_a \phi) = 0.$$

But the left-hand side is independent of x_{2n} and thus retains the same form whatever value be assigned to x_{2n}. When the value zero is assigned to it, x becomes a; and the equation is

$$\frac{1}{N(a)} X(a)\, da = 0$$

or removing the numerical factor $N(a)$ and writing A for $X(a)$ we have it

$$A da = 0.$$

The quantity a involves only $2n-1$ of the units; and thus the equation has been transformed and it involves only $2n-1$ numerical variables.

* This is a point of weakness in Grassmann's method regarded in any light other than most purely theoretical; for the form of the function is an infinite series, arising from a reversion of a Taylor's series, and there is no proof that the series converges. Even if the series converge, the form of the integral so found is such as to render this stage of the method unpractical; and so it must remain until some other process is devised for the integration of a differential equation with its dependent variable an extensive quantity.

It is easy to see that the extensive equation

$$a = \text{function of } x \text{ and } x_{2n}$$

leads to $2n-1$ numerical equations of the form

$$a_i = \text{function of } x_1, \ldots\ldots, x_{2n},$$

so that these $2n-1$ equations are equivalent to a set of integrals of the ordinary subsidiary system. Their form corresponds to the form in which Jacobi (§ 70) took the integral system, viz., in the introduction of the principal integrals, a correspondence directly intended by Grassmann.

82. We now take $m > 2n$; we have

$$\left[X\left(\frac{dX}{dx}\right)^n\right] = 0, \qquad \left[X\left(\frac{dX}{dx}\right)^{n-1}\right] \gtrless 0.$$

The characteristic equation which, if satisfied, makes the transformation of Xdx possible is

$$\left[\frac{dX}{dx}\, c d_t x\right] = \lambda X c,$$

where c is of the same species as x. Let $c = e_1, \ldots, e_m$ in turn, and write

$$G_s = \left[\frac{dX}{dx}\, e_s d_t x\right] - \lambda X e_s;$$

then the characteristic equation is replaced by

$$G_1 = G_2 = \ldots = G_m = 0.$$

First we shall assume that $\left[\left(\frac{dX}{dx}\right)^n\right]$ does not vanish and therefore also that λ does not vanish.

Let $e_1, \ldots, e_{2n}, e_r$ be $2n+1$ of the m units, for $r = 2n+1, \ldots, m$; let E denote $[e_1 \ldots e_{2n}\, e_r]$ and take F_s as the product of all the e's in E except e_s, so that $[e_s F_s] = E$; and finally let a_s denote the numerical expression

$$\left[\left(\frac{dX}{dx}\right)^n F_s\right].$$

Then

$$\Sigma a_s G_s = \Sigma \left[\left(\frac{dX}{dx}\right)^n F_s\right]\left[\frac{dX}{dx}\, e_s d_t x\right] - \lambda \Sigma X e_s \left[\left(\frac{dX}{dx}\right)^n F_s\right]$$

$$= -(2n+1)\left[\left(\frac{dX}{dx}\right)^{n+1} E d_t x\right] - \lambda(2n+1)\left[X\left(\frac{dX}{dx}\right)^n E\right].$$

But we have

$$\left[X\left(\frac{dX}{dx}\right)^n\right]=0, \text{ and } \left[\left(\frac{dX}{dx}\right)^{n+1}\right]=0;$$

hence the right-hand side vanishes, and therefore

$$\Sigma a_s G_s = 0.$$

Whence it follows that there is a numerical relation between $2n+1$ of the equations $G=0$; and by varying the choice of units that there is a relation between every $2n+1$ of those equations.

It may happen that for some selection of $2n+1$ units some of the quantities a may be zero; this however cannot happen for all the selections, otherwise $\left[\left(\frac{dX}{dx}\right)^n\right]$ would vanish, contrary to hypothesis. Hence some of the equations must contain $2n+1$ terms G; and therefore we can choose a set of $2n$ of the quantities $G_1, \ldots, G_m$, say $G_1, \ldots, G_{2n}$, such that the remaining $m-2n$ are numerically derivable from that set.

We thus have only $2n$ numerical equations, independent of one another and involving the quantities $d_tx_1, d_tx_2, \ldots, d_tx_m$ and λ. As before, we replace λ by a corresponding quantity μ; the equations will be solved so as to give quantities d_tx_i. As there are only $2n$ equations, only $2n$ of the quantities can be determined definitely; those associated with the remainder will be taken to be zero. Let those so determined be $d_tx_1, \ldots, d_tx_{2n}$; then $x_{2n+1}, \ldots, x_m$ are regarded as constants. The $2n$ equations are now equations in μ and $d_tx_1, \ldots, d_tx_{2n}$; when treated in the same way as the set in § 80, they lead to

$$d_t(e_1x_1+\ldots+e_{2n}x_{2n})=\mu\left[X\left(\frac{dX}{dx}\right)^{n-1}e_1\ldots e_{2n}\right],$$

and therefore, as $d_tx_r=0$ for $r=2n+1, \ldots, m$, we have

$$d_tx=\mu\left[X\left(\frac{dX}{dx}\right)^{n-1}e_1\ldots e_{2n}\right].$$

This equation is integrated as before and the integral involves an extensive constant a, which is the value of x when t (taken to be x_{2n}) is zero; and substitution leads to an equation of the form

$$A\,da=0,$$

where A is the same function of a as X is of x, and the new quantity a involves only $m-1$ units viz., $e_1, \ldots, e_{2n-1}, e_{2n+1}, \ldots, e_m$.

It was assumed that $\left[\left(\frac{dX}{dx}\right)^n\right]$ is different from zero. When the alternative holds, viz., when

$$\left[\left(\frac{dX}{dx}\right)^n\right] = 0,$$

so that $\lambda = 0$ and therefore N is independent of t, then as in § 79 we prove that

$$d_t x = \mu\left[X\left(\frac{dX}{dx}\right)^{n-1} e_1 \ldots e_{2n}\right],$$

which is the value in the preceding case, is still correct and is sufficient to effect the required transformation. For with this value we have, for $s = 1, \ldots, m$,

$$\begin{aligned}\left[\frac{dX}{dx} e_s d_t x\right] &= \mu\left[\frac{dX}{dx} e_s \left[X\left(\frac{dX}{dx}\right)^{n-1} e_1 \ldots e_{2n}\right]\right] \\ &= -\frac{\mu}{2n}\sum_{i=1}^{2n} X e_i \left[\left(\frac{dX}{dx}\right)^n e_s E_i\right] \\ &= 0\end{aligned}$$

under the present supposition. As this is true for each of the m units, we have, on constructing $d_a x$, the equation

$$\left[\frac{dX}{dx} d_a x d_t x\right] = 0,$$

and therefore

$$\frac{dX}{dx} d_a x d_t x - \frac{dX}{dx} d_t x d_a x = 0.$$

Also, as in the earlier case,

$$X d_t x = 0;$$

from which point the rest of the proof is as before, leading to the result that

$$X d_a x = 0$$

is independent of t and dt and is expressed by means of $m-1$ units.

Hence *by the transformation derived from the integration of the equation*

$$d_t x = \mu\left[X\left(\frac{dX}{dx}\right)^{n-1} e_1 \ldots e_{2n}\right]$$

we pass from the equation $Xdx = 0$, where x is an extensive magnitude involving m units ($m > 2n$), to

$$Ada = 0,$$

where a is an extensive magnitude involving $m-1$ units and A is the same function of a as X is of x.

83. The equation

$$\left[X\left(\frac{dX}{dx}\right)^n\right] = 0$$

is identically satisfied whatever be the value of x. If then we replace x by a and X by A, the same function of a as X is of x, we have

$$\left[A\left(\frac{dA}{da}\right)^n\right] = 0.$$

If now $m-1$, the number of units in a, be greater than $2n$, we can use the foregoing method to transform $Ada = 0$ into $Bdb = 0$, where b involves $(m-1)-1 = m-2$ units and B is the same function of b as A is of a and therefore as X is of x; and we have

$$\left[B\left(\frac{dB}{db}\right)^n\right] = 0.$$

Proceeding in this manner we can reduce the equation until the extensive variable k involves only $2n$ units: and it takes the form

$$Kdk = 0,$$

where K is the same function of k as X is of x; and this reduction has been effected under the supposition that

$$\left[X\left(\frac{dX}{dx}\right)^n\right] = 0,$$

conditions which are necessary and sufficient.

Finally, applying to this equation the process of § 81 we come to an equation of the form

$$Ydy = 0,$$

where y involves $2n-1$ units only and Y is the same function of y as X is of x.

84. The rest of Grassmann's reduction of $Xdx = 0$ to the form $\sum_{i=1}^{n} U_i du_i = 0$ proceeds as in the Pfaffian method. One of the numerical variables in y is made constant, so that the equation $Ydy = 0$ then involves only $2n-2$ numerical variables and there-

fore $2n-2$ units. The reduction of the modified Ydy to involve only $2n-3$ units is effected as in § 81; and then one of the new numerical variables is made constant, so that the equation now involves $2n-4$ numerical variables and therefore $2n-4$ units: and so on until the end. Evidently n reductions are necessary, each reduction furnishing a numerical variable made a constant and therefore furnishing an integral, so that n integrals are obtained; and then the formation of the expression ΣUdu, if desired, is only a matter of comparison of coefficients.

Ex. 1. Denoting by $[1, \ldots, n]_{2r}$ the system

$$\left\| \begin{matrix} \dfrac{\partial}{\partial x_1}, & \dfrac{\partial}{\partial x_2}, \ldots\ldots, & \dfrac{\partial}{\partial x_n} \\ X_1, & X_2, \ldots\ldots, & X_n \\ \dfrac{\partial}{\partial x_1}, & \dfrac{\partial}{\partial x_2}, \ldots\ldots, & \dfrac{\partial}{\partial x_n} \\ X_1, & X_2, \ldots\ldots, & X_n \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \end{matrix} \right\|,$$

where there are $2r$ rows and in the expansion of a determinant the i^{th} factor of any term is to be taken from the i^{th} row, shew that $[1, \ldots, n]_{2r}=0$ is the condition that

$$\sum_{i=1}^{n} X_i dx_i$$

should be reducible to the form

$$du_1 + \sum_{i=2}^{r} v_i du_i. \qquad \text{(Tanner.)}$$

Ex. 2. Similarly denoting by $[1, \ldots\ldots, n]_{2r+1}$ the system

$$\left\| \begin{matrix} X_1, & X_2, \ldots\ldots, & X_n \\ \dfrac{\partial}{\partial x_1}, & \dfrac{\partial}{\partial x_2}, \ldots\ldots, & \dfrac{\partial}{\partial x_n} \\ X_1, & X_2, \ldots\ldots, & X_n \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \end{matrix} \right\|,$$

where there are $2r+1$ rows and the law of expansion is as before, shew that $[1, \ldots, n]_{2r+1}=0$ is the condition that

$$\sum_{i=1}^{n} X_i dx_i$$

should be reducible to the form

$$\sum_{i=1}^{r} v_i du_i. \qquad \text{(Tanner.)*}$$

* These results are to be found in Prof. Tanner's memoirs "On the transformation of a linear differential expression," *Quart. Math. Journ.* vol. xvi. (1879) pp. 45—64. Their form will be seen to have a close resemblance to that of Grassmann's conditions (§ 73), which is the reason for placing Prof. Tanner's results in this connection.

CHAPTER VI.

NATANI'S METHOD.

85. IN order to transform the differential expression in

$$X_1 dx_1 + X_2 dx_2 + X_3 dx_3 + \ldots\ldots + X_n dx_n = 0 \ldots\ldots\ldots\ldots\text{(I)}$$

so that it may contain a smaller number of differential elements, Natani* introduces a set of n functions of the variables $t_1, t_2, \ldots, t_p, u_1, u_2, \ldots, u_q$, where $p+q=n$, which in the first place satisfy the single condition of being functionally independent of one another. Thus

$$\sum_{m=1}^{n} X_m \delta x_m = \sum_{r=1}^{p} T_r \delta t_r + \sum_{s=1}^{q} U_s \delta u_s,$$

where the variations of the variables x are any whatever; and the coefficients T and U are given by

$$T_r = \sum_{m=1}^{n} X_m \frac{\partial x_m}{\partial t_r}, \quad U_s = \sum_{m=1}^{n} X_m \frac{\partial x_m}{\partial u_s}.$$

When the variations of the variables x are such as to have (I) satisfied, the consequent variations of t and u are subject to the relation

$$\sum_{r=1}^{p} T_r dt_r + \sum_{s=1}^{q} U_s du_s = 0.$$

This relation can be satisfied by simultaneous equations of two kinds: (α) those formed by equating the differential elements to zero, and (β) those formed by equating the coefficients of differential elements to zero. An equation of the kind (α) implies that

[1] *Crelle*, t. lviii. (1861) pp. 301—328; and Natani, *Die höhere Analysis* (1866), pp. 304—327.

the variable with the vanishing differential element is unchanging; and we should thence have

$$\text{function } (x_1, x_2, \ldots, x_n) = \text{constant}$$

as a relation which is used to satisfy (I) and which is therefore an integral of the differential equation. An equation of the kind (β) leads to a partial differential equation of the form

$$\sum_{m=1}^{n} X_m \frac{\partial x_m}{\partial z} = 0,$$

where dz is one of the non-vanishing differential elements. And we consider included in (α) any homogeneous relation among the new differential elements, which leads to an equation

$$\text{function } (t_1, t_2, \ldots, t_p, u_1, \ldots, u_q) = \text{constant};$$

because the substitution of their values for $t_1, \ldots, t_p, u_1, \ldots, u_q$ leads to an integral of (I) as just explained.

86. We take then all the new variables with vanishing differential elements to be the variables u, and all those, the coefficients of the differential elements of which vanish, to be the variables t.

From the first of the suppositions each of the quantities u is an integral of the equation (I).

From the second supposition there are p equations of the form

$$\sum_{m=1}^{n} X_m \frac{\partial x_m}{\partial t_r} = 0 \quad \text{.........................(II)}$$

for $r = 1, 2, \ldots, p$. In the initial substitution the quantities $t_1, \ldots, t_p$ were functionally independent of one another; the transformation of the differential equation has given no relation among their variations; and hence they may be looked upon as p independent variables. Moreover because the equation is satisfied whatever be the variations of the variables t, it follows that the *variables t may be taken to be quite arbitrary quantities* subject to the single condition that no identical functional relations among $t_1, \ldots, t_p, u_1, \ldots, u_q$ exist. Since then we have

$$\sum_{m=1}^{n} X_m dx_m = 0,$$

and $t_1, t_2, \ldots, t_p$ are the only independent variables, we have

$$\sum_{m=1}^{n} X_m \frac{\partial x_m}{\partial t_r} = 0$$

for $r = 1, 2, \ldots, p$, which is the system (II); hence the system (II) and the equation (I) are co-extensive* and therefore $u_1, \ldots, u_q$ which are integrals of (I) are also integrals of (II).

87. Thus there will be ultimately a set of integral equations and a set of independent variables. That aggregate of equations and independent variables will be looked upon as the most general combination, which contains the greatest number of independent variables and consequently the smallest number of integral equations; for in that combination the variations of the variable quantities will be the least limited. Hence the most general solution of the equation (I) will be obtained by making q a minimum.

Now, since we have

$$\sum_{m=1}^{n} X_m \delta x_m = \sum_{s=1}^{q} U_s \delta u_s \qquad \text{(III)}$$

after the preceding inferences, it follows that

$$X_m = \sum_{s=1}^{q} U_s \frac{\partial u_s}{\partial x_m} \qquad \text{(III}^a\text{)},$$

a system of equations n in number; they may be regarded as determining the (as yet) unknown quantities u and U, which are $2q$ in number.

If n be greater than $2q$, so that there are more equations than unknown quantities to be determined, the elimination of U and u from the system (III^a) will lead to relations among the quantities X. Such relations we shall at present assume not to exist; and hence $2q$, which is to be a minimum, must be chosen so as to be not less than n.

When n *is even*, the equations (III^a) can be considered as determining $q\,(=\tfrac{1}{2}n)$ quantities u and $q\,(=\tfrac{1}{2}n)$ quantities U; and thus we have, for the most general case of unconditioned coefficients,

$$\sum_{m=1}^{2n} X_m \delta x_m = \sum_{s=1}^{n} U_s \delta u_s \qquad \text{(1)}.$$

* For a similar result in the theory of systems of exact equations, see § 22.

When *n is odd*, we have (since $2q$ may not be less than n) the smallest value of q given by

$$2q = n + 1.$$

There are thus $\frac{1}{2}(n+1)$ coefficients U and $\frac{1}{2}(n+1)$ variables u to be determined from only n equations; and so we may have, either

(A) $\frac{1}{2}(n+1)$ coefficients U and $\frac{1}{2}(n-1)$ variables u determined and the remaining variable left undetermined;

or

(B) $\frac{1}{2}(n+1)$ variables u and $\frac{1}{2}(n-1)$ coefficients U determined and the remaining coefficient left undetermined. But in (A) the undetermined variable cannot be an absolute constant, for otherwise the corresponding term will not occur and there will only be $n-1$ quantities U and u given by the n equations, implying that there is one relation among the quantities X: and in (B) the coefficient may not be taken to be zero, for otherwise the corresponding term will not occur—on account of the same unjustified inference as before.

Since that final arrangement among the variables is being sought which contains the smallest number of integral relations, we choose (A) in preference to (B) as being slightly more general: it has $\frac{1}{2}(n-1)$ determined integral relations and one which is arbitrary, while (B) has $\frac{1}{2}(n+1)$ determined integral relations.

Let ϕ be the arbitrary variable left undetermined in (A); then we have

$$\sum_{m=1}^{2n+1} X_m \delta x_m = \lambda \delta\phi + \sum_{s=1}^{n} U_s \delta u_s \quad \text{............(2)},$$

where $\lambda, U_1, \ldots, U_n, u_1, \ldots, u_n$ are the $2n+1$ quantities determined by the associated $2n+1$ equations of the type (III^a).

88. We consider first the case, in which the number of variables in equation (I) is originally even; we then have

$$\Omega = \sum_{m=1}^{2n} X_m \delta x_m = \sum_{s=1}^{n} U_s \delta u_s \quad \text{...............(1)},$$

where the coefficients U and the n new variables u are functions of the variables $x_1, \ldots, x_{2n}$ and independent of one another. The integrals of the equation $\Omega = 0$ are

$$u_1 = \text{constant}, \ \ldots\ldots\ldots\ldots\ldots\ldots, \ u_n = \text{constant};$$

that is, n of the new variables give integrals of the equations, and the remaining n of the new variables $t_1, \ldots, t_n$ do not explicitly occur in the integrals. And we have seen that the variables t are undetermined and are arbitrary, subject to the single condition that no functional relations subsist among the quantities u and t.

Thus the variables u are explicitly independent of the variables t and at the same time are integrals of the equations

$$\sum_{m=1}^{2n} X_m \frac{\partial x_m}{\partial t_r} = 0$$

for $r = 1, 2, \ldots, n$. The manner in which this result exists is as follows: From the equation $u_s =$ constant we have

$$0 = \delta u_s = \sum_{m=1}^{2n} \frac{\partial u_s}{\partial x_m} \delta x_m;$$

and, since $t_1, \ldots, t_n$ are the independent variables, we have

$$\delta x_m = \sum_{r=1}^{n} \frac{\partial x_m}{\partial t_r} \delta t_r,$$

so that

$$0 = \sum_{m=1}^{2n} \frac{\partial u_s}{\partial x_m} \sum_{r=1}^{n} \frac{\partial x_m}{\partial t_r} \delta t_r,$$

or, since all the variations δt_r are independent of one another, we have

$$0 = \sum_{m=1}^{2n} \frac{\partial u_s}{\partial x_m} \frac{\partial x_m}{\partial t_r}$$

for $s = 1, 2, \ldots, n$ and $r = 1, 2, \ldots, n$. From this last set of equations we can find uniquely, for each of the values of r, expressions for $\frac{\partial x_1}{\partial t}, \frac{\partial x_2}{\partial t}, \ldots, \frac{\partial x_n}{\partial t}$ which are linear in $\frac{\partial x_{n+1}}{\partial t}, \ldots, \frac{\partial x_{2n}}{\partial t}$; when these are substituted in

$$\sum_{m=1}^{2n} X_m \frac{\partial x_m}{\partial t},$$

the resulting expression vanishes identically whatever be the values of $\frac{\partial x_{n+1}}{\partial t}, \ldots, \frac{\partial x_{2n}}{\partial t}$.

Thus $u_1, \ldots, u_n$ are solutions of the system of equations

$$\sum_{m=1}^{2n} X_m \frac{\partial x_m}{\partial t_r} = 0 \quad \text{.......................(3)};$$

and, in particular, since $t_2, t_3, \ldots, t_n$ are independent of t_1, we have $u_1, \ldots, u_n, t_2, \ldots, t_n$ as solutions of

$$\sum_{m=1}^{2n} X_m \frac{\partial x_m}{\partial t_1} = 0.$$

89. So far, the question has been regarded as one of transformation of Ω to a form which shall contain the smallest possible number of differential elements; in order to obtain the integral equivalent of $\Omega = 0$ it is necessary to have the explicit forms of $u_1, \ldots, u_n$. These (being independent of $t_1, \ldots, t_n$) are unaffected by particular forms properly assigned to $t_1, \ldots, t_n$; and so we assume, in conformity with Pfaff's result (§ 60), that t_1 enters into Ω only as a factor V common to all the coefficients of the differential elements. This assumption, repeated for successive reductions, might be expressed by taking (as a possible way)

$$U_1 = V\alpha_1,\ U_2 = V\alpha_2, \ldots\ldots, U_n = V\alpha_n$$

with

$$V = \frac{1}{t_1},\ \alpha_1 = 1,\ \alpha_2 = \frac{1}{t_2},\ \alpha_3 = \frac{1}{t_2 t_3}, \ldots\ldots, \alpha_n = \frac{1}{t_2 t_3 \ldots t_n},$$

which are consistent with all the inferences which have been made and with all the conditions which subsist; but this form of coefficients is of course not unique.

Replacing t_1 by A we have

$$\sum_{m=1}^{2n} A X_m \delta x_m = \sum_{s=1}^{n} \alpha_s \delta u_s \quad \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

where the right-hand side is independent of t_1; and the equation, which has $t_2, \ldots, t_n, u_1, \ldots, u_n$ for solutions, may be written

$$\sum_{m=1}^{2n} A X_m \frac{\partial x_m}{\partial t_1} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

which is identically satisfied when the proper values of x in terms of t and other variables are substituted.

Take now any arbitrary variations of the variables*; then we

* In this idea of two independent sets of variations of the variables Natani was anticipated by Binet, who, in his memoir "Sur la transformation de Pfaff relative aux fonctions différentielles linéaires contenant un nombre pair de variables," *Comptes Rendus*, t. xv. (1842), pp. 74—80, had applied it to the case of an even unconditioned Pfaffian. The association of initial values of the variables with the equation is also discussed in this memoir; and the transformation to the form

have, in virtue of the solutions of equation (5),

$$\delta\left\{\sum_{m=1}^{2n} AX_m \frac{\partial x_m}{\partial t_1}\right\} = 0,$$

or

$$A\sum_{m=1}^{2n}\frac{\partial x_m}{\partial t_1}\delta X_m + \sum_{m=1}^{2n} AX_m\delta\frac{\partial x_m}{\partial t_1} + \left(\sum_{m=1}^{2n} X_m\frac{\partial x_m}{\partial t_1}\right)\delta A = 0,$$

which, since the last term vanishes, gives

$$A\sum_{m=1}^{2n}\frac{\partial x_m}{\partial t_1}\delta X_m + \sum_{m=1}^{2n} AX_m\delta\frac{\partial x_m}{\partial t_1} = 0.$$

But, since $\sum_{s=1}^{n} \alpha_s \delta u_s$ is independent of t_1, we have by (4)

$$\frac{\partial}{\partial t_1}\left(\sum_{m=1}^{2n} AX_m\delta x_m\right) = 0,$$

or

$$\sum_{m=1}^{2n}\frac{\partial}{\partial t_1}(AX_m)\,\delta x_m + \sum_{m=1}^{2n} AX_m\frac{\partial}{\partial t_1}\delta x_m = 0.$$

Now δx_m is an arbitrary variation of x_m, so that

$$\frac{\partial}{\partial t_1}\delta x_m = \delta\frac{\partial x_m}{\partial t_1};$$

and therefore from the last two equations we have

$$A\sum_{m=1}^{2n}\frac{\partial x_m}{\partial t_1}\delta X_m = \sum_{m=1}^{2n}\frac{\partial}{\partial t_1}(AX_m)\,\delta x_m.$$

And

$$\delta X_m = \sum_{s=1}^{2n}\frac{\partial X_m}{\partial x_s}\delta x_s,$$

where the variations δx are arbitrary, so that coefficients of δx must be equal on the two sides of the resulting equation; hence from the coefficients of δx_s we have

$$\begin{aligned} A\sum_{m=1}^{2n}\frac{\partial x_m}{\partial t_1}\frac{\partial X_m}{\partial x_s} &= \frac{\partial}{\partial t_1}(AX_s) \\ &= X_s + A\sum_{m=1}^{2n}\frac{\partial X_s}{\partial x_m}\frac{\partial x_m}{\partial t_1} \end{aligned}$$

(4^a) (see § 91 post) is also indicated. Binet however limited himself to this result; his object was to indicate a process of transformation new at the time of publication of his memoir.

for $A = t_1$; and therefore

$$X_s = A \sum_{m=1}^{2n} a_{m,s} \frac{\partial x_m}{\partial t_1} = t_1 \sum_{m=1}^{2n} a_{m,s} \frac{\partial x_m}{\partial t_1} \quad \ldots\ldots\ldots\ldots(6);$$

and this holds for $s = 1, 2, \ldots, 2n$. It is a system of equations satisfied in virtue of the solutions of equation (5); it involves the $2n$ quantities $t_1 \dfrac{\partial x_m}{\partial t_1}$ and thus a complete integral of the system is constituted by $u_1, \ldots, u_n, t_2, \ldots, t_n$ (which are all explicitly independent of t_1) and the equation

$$\log t_1 = \int \sum_{m=1}^{2n} \left(a_{m,s} \frac{dx_m}{X_s} \right).$$

The last equation is unnecessary for our immediate purpose—the deduction of the quantities u; hence we only consider the $2n-1$ other integrals. From them the rejection of $t_2, \ldots, t_n$ must be made: and this can be effected by other differential equations of the type (5).

90. The $2n-1$ retained integrals of the system (6) will not necessarily occur in the forms $u_1, \ldots, u_n, t_2, \ldots, t_n$; let them be $\beta_1, \beta_2, \ldots, \beta_{2n-1}$, so that all the quantities u and t—and hence also all the coefficients a in (4)—can be expressed in terms of the quantities β. Substituting them in the right-hand side of (4) we have

$$\sum_{m=1}^{2n} A X_m \delta x_m = \sum_{s=1}^{2n-1} B_s \delta \beta_s,$$

and therefore the differential equation (I) is replaced by

$$\sum_{s=1}^{2n-1} B_s d\beta_s = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

where all the coefficients B are functions of the variables β alone. It has thus become an equation in an odd number of variables, and therefore (§ 69) one of its integrals is arbitrary, say

$$\phi(\beta_1, \beta_2, \ldots, \beta_{2n-1}).$$

This without any loss of generality may be taken to be β_1, for every integral of one system can be expressed in terms of the integrals of another equivalent system. Hence we take β_1 as an integral; since it is of the form

$$u_1 = \beta_1 = \text{constant},$$

the equation (7) is, by the use of this integral, reduced to contain only $2n-2$ variables, that is, an even (and a diminished) number of variables, after one of the integrals u_1 of the original equation has been obtained.

The foregoing process may now be re-applied; each successive stage diminishes the number of variables by two and provides one of the integrals u of the original equation. Hence finally we shall have, after n applications, the set of integrals $u_1, \ldots, u_n$ of the equation.

91. The actual expression for the transformed value of Ω depends upon the choice of the individual members of the system of integrals of the equations (6). In order to have the expression simple, Natani chooses the *principal integrals* (§ 70) for this system. Taking any set of integrals, say the system of the quantities β, we have them in the form

$$\beta_r(x_1, x_2, \ldots, x_{2n}) = \text{constant} = \beta_r,$$

the functions β_r being known; the right-hand side is unaltered, if we assign any particular value to x_1 and the corresponding values to $x_2, \ldots, x_{2n}$. Let the particular value of x_1 be zero (or, if this be inconvenient, a constant) and let the values of the remaining variables be $x_2', \ldots, x'_{2n}$; then we have

$$\beta_r(0, x_2', x_3', \ldots, x_m') = \text{constant} = \beta_r.$$

This is a system of $2n-1$ equations in $2n-1$ quantities x', and the equations are independent of one another; they therefore give these quantities x' as independent functions of the β's and consequently as a system of integrals of the equations (6).

Introducing these quantities x' as integrals into the equation (4) and bearing in mind that the u's and the α's are functions of them, we have

$$\sum_{m=1}^{2n} AX_m \delta x_m = \sum_{m=2}^{2n} K_m \delta x_m',$$

where the coefficients K are functions of the variables x' only. The equation just obtained holds for all values of $x_1, x_2, \ldots, x_{2n}$ and therefore for

$$x_1 = 0 \text{ (or the constant value)}, \; x_2 = x_2', \ldots\ldots, x_{2n} = x'_{2n};$$

hence, if A' and X' be the values of A and X on the substitution of these values, we have

$$\sum_{m=2}^{2n} A'X_m'\delta x_m' = \sum_{m=2}^{2n} K_m \delta x_m';$$

the term on the left-hand side, which corresponds to $m=1$, no longer occurring. Since the quantities x_m' are functionally independent and the variations are arbitrary, we have

$$K_m = A'X_m',$$

so that, when $x_2', \ldots, x'_{2n}$ are known, the coefficients in the transformed value of Ω are, save as to a factor, determined by inspection; and the result is

$$\Omega = \sum_{m=1}^{2n} X_m \delta x_m = \frac{A'}{A} \sum_{m=2}^{2n} X_m' \delta x_m' \quad \ldots\ldots\ldots\ldots(4^a).$$

Now since $x_2', \ldots, x'_{2n}$ is a system of integrals of (6), the first of the integrals of the differential equation $\Omega = 0$ is, as in § 90, given by

$$x_2' = \text{constant} = c_1;$$

and the equation to be integrated is now

$$\Omega_1 = \sum_{m=3}^{2n} X_m' dx_m' = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{I}_1),$$

containing only $2n-2$ variables.

We proceed in the same manner with (I_1). The system which corresponds to (6) is constructed and integrated, with a result of the form

$$\gamma_r(x_3', x_4', \ldots, x'_{2n}) = \text{constant},$$

where $r = 1, 2, \ldots, 2n-3$; and introducing the principal integrals of the new system by taking $x_3' = 0$ (or, if this be inconvenient, a constant) and $x_4' = x_4'', \ldots, x'_{2n} = x''_{2n}$, the system of principal integrals $x_4'', \ldots, x''_{2n}$ is given by the $2n-3$ equations

$$\gamma_r(x_3', x_4', \ldots, x'_{2n}) = \gamma_r(0, x_4'', \ldots, x''_{2n}),$$

where the function γ_r is known. As before, these lead to a transformation

$$\sum_{m=3}^{2n} X_m' \delta x_m' = \frac{A_3''}{A_3} \sum_{m=4}^{2n} X_m'' \delta x_m'',$$

where X_m'' is derived from X_m' by substituting $x_3' = 0, x_4' = x_4'', \ldots$ $x'_{2n} = x''_{2n}$ i.e., from X_m by substituting $x_1 = 0$, $x_2 = c_1$, $x_3 = 0$,

$x_4 = x_4''$, $x_5 = x_5''$, ..., $x_{2n} = x''_{2n}$ and so is given by inspection. The first of the integrals of $\Omega_1 = 0$ (and therefore the second of the integrals of $\Omega = 0$) is, as before, given by

$$x_4'' = \text{constant} = c_2;$$

and the equation to be integrated is now

$$\Omega_2 = \sum_{m=5}^{2n} X_m'' dx_m'' = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (\mathrm{I}_2).$$

The process may now be similarly applied to Ω_2; and so on in succession, until finally we obtain the system of n integrals of $\Omega = 0$ in the form

$$x_2' = c_1,\ x_4'' = c_2,\ x_6''' = c_3,\ \ldots\ldots,\ x_{2n}^{(n)} = c_n.$$

At each step the quantity A can be obtained, as in § 89, by a single quadrature; and the resulting final form of the transformation of Ω is evidently

$$\sum_{m=1}^{2n} X_m \delta x_m = \frac{A_1'}{A_1} X_2' \delta x_2' + \frac{A_1' A_2''}{A_1 A_2} X_4'' \delta x_4'' + \frac{A_1' A_2'' A_3'''}{A_1 A_2 A_3} X_6''' \delta x_6''' + \ldots$$
$$+ \frac{A_1' A_2'' A_3''' \ldots A_n^{(n)}}{A_1 A_2 A_3 \ldots A_n} X_{2n}^{(n)} \delta x_{2n}^{(n)}.$$

The coefficients X_2', X_4'', ... are derivable from X_2, X_4, ... by inspection, when the values of the quantities x', x'', x''', ... are known. And these values are given by the integration of the subsidiary systems (6) which are n in number, there being one such system for each reduction of the number of variables in the quantities Ω.

It is hardly necessary to point out that the system (6) is essentially the same as the first form (8) of § 55 of the subsidiary equations in Pfaff's reduction, and it can therefore be replaced by the equivalent subsidiary system (14) of § 59 resulting from the solution of the equations considered as a system linear in the derivatives of x_1, x_2, ...

92. Considerable simplification arises in any case in which a number, say $n - p$, of the original coefficients X vanish*; let them be X_{n+p+1}, X_{n+p+2}, ..., X_{2n}, the remaining non-vanishing coefficients being supposed to be functions of the variables x_1, x_2, ..., x_{2n}.

* See also Jacobi, *Ges. Werke*, t. iv. p. 125.

When p integrals have been obtained, consequent on the integration of p of the subsidiary systems, in the form

$$x_2' = c_1, \; x_4'' = c_2, \ldots, \; x_{2p}^{(p)} = c_p,$$

the equation remaining to be integrated is

$$X_{2p+1}^{(p)} \, dx_{2p+1}^{(p)} + X_{2p+2}^{(p)} \, dx_{2p+2}^{(p)} + \ldots + X_{n+p}^{(p)} \, dx_{n+p}^{(p)} = 0.$$

For the original equation $n-p$ integrals are still necessary; but these are given by

$$x_{2p+1}^{(p)} = c_{p+1}, \; x_{2p+2}^{(p)} = c_{p+2}, \ldots, \; x_{n+p}^{(p)} = c_n,$$

so that only p integrations of subsidiary systems are necessary.

This is of especial importance in the integration of partial differential equations of the first order; for, if the equation to be integrated be

$$p_n = \phi = \phi(z, x_1, \ldots, x_n, p_1, \ldots, p_{n-1}),$$

we have

$$dz - p_1 dx_1 - \ldots - p_{n-1} \, dx_{n-1} - \phi dx_n$$
$$- 0 \, . \, dp_1 - 0 \, . \, dp_2 - \ldots - 0 \, . \, dp_{n-1} = 0,$$

so that $n-1$ of the coefficients X vanish and therefore only a single integration of a subsidiary system is necessary*.

Ex. 1. As an example of Natani's process, consider

$$\Omega = x_2 dx_1 + x_3 dx_2 + x_4 dx_3 + x_5 dx_4 + x_6 dx_5 + x_1 dx_6 = 0.$$

When we form the equations (6), they are

$$x_2 = \frac{t_1}{dt_1}(-dx_2 + dx_6), \qquad x_3 = \frac{t_1}{dt_1}(dx_1 - dx_3),$$

$$x_4 = \frac{t_1}{dt_1}(dx_2 - dx_4), \qquad x_5 = \frac{t_1}{dt_1}(dx_3 - dx_5),$$

$$x_6 = \frac{t_1}{dt_1}(dx_4 - dx_6), \qquad x_1 = \frac{t_1}{dt_1}(-dx_1 + dx_5),$$

from which it follows that $dt_1 = 0$ and that $dx_1 = dx_3 = dx_5$, $dx_2 = dx_4 = dx_6$. From the former we have

$$t_1 = \text{constant},$$

or

$$A = \text{constant};$$

and, since A does not involve the variables, the value of A, viz. A', when the principal integrals are substituted is unaltered, and therefore $\frac{A'}{A} = 1$.

* See Chap. VII.

The other equations however are insufficient in their present form. The present example is an illustration of § 62, for it is easy to verify that

$$[123456]=0,$$

while non-vanishing Pfaffians of next lower order are

$$[1234]=1,\ [1236]=-1,\ [1245]=1,\ [1256]=1,\ [1346]=-1,$$
$$[1456]=-1,\ [2345]=1,\ [2356]=1,\ [3456]=1.$$

The theorem of § 62 therefore applies: we easily find

$$W_1=W_3=W_5=x_1+x_3+x_5=P,$$
$$W_2=W_4=W_6=x_2+x_4+x_6=Q;$$

and the subsidiary equations are

$$\frac{dx_1}{P}=\frac{dx_3}{P}=\frac{dx_5}{P}=-\frac{dx_2}{Q}=-\frac{dx_4}{Q}=-\frac{dx_6}{Q}.$$

Five independent integrals of these equations are

$$\left.\begin{aligned} x_3-x_1&=\text{constant}=x_3' \\ x_5-x_1&=\text{constant}=x_5' \\ x_4-x_2&=\text{constant}=x_4'-x_2' \\ x_6-x_2&=\text{constant}=x_6'-x_2' \\ (x_1+x_3+x_5)(x_2+x_4+x_6)&=\text{constant}=(x_3'+x_5')(x_2'+x_4'+x_6') \end{aligned}\right\},$$

with the introduction of Natani's variables: they determine x_2', x_3', x_4', x_5', x_6' as functions of the original variables.

Since $\dfrac{A'}{A}$ is unity, we have

$$x_2\delta x_1+x_3\delta x_2+x_4\delta x_3+x_5\delta x_4+x_6\delta x_5+x_1\delta x_6=x_3'\delta x_2'+x_4'\delta x_3'+x_5'\delta x_4'+x_6'\delta x_5'$$

by the general theory: there are only four differential elements but there are five variables on the right-hand side, an illustration of § 92. The first integral of $\Omega=0$ is therefore taken in the form

$$x_2'=\text{constant}=c_1;$$

and the equation now to be integrated is

$$x_4'dx_3'+x_5'dx_4'+x_6'dx_5'=0.$$

Though it is easy to see at once what the integral system of this equation is, the application of the general rule is of interest. The equations (6) of § 89—or, their equivalent, the subsidiary system of § 62—give

$$\frac{dt_1'}{t_1'}=\frac{dx_3'}{x_5'}=\frac{dx_4'}{-x_4'}=\frac{dx_5'}{0}=\frac{dx_6'}{-x_4'-x_6'}.$$

First, we have

$$A_2=t_1'=\frac{\text{constant}}{x_4'},$$

and therefore, in accordance with Natani's process,

$$\frac{A_2''}{A_2}=\frac{x_4'}{x_4''}.$$

Three independent integrals of the subsidiary system, not involving t_1', are

$$\left.\begin{aligned} x_5' &= \text{constant} = x_5'' \\ x_4' e^{\frac{x_3'}{x_5'}} &= \text{constant} = x_4'' \\ x_4' e^{-\frac{x_6'}{x_4'}} &= \text{constant} = x_4'' e^{-\frac{x_6''}{x_4''}} \end{aligned}\right\},$$

when we take $x_3' = 0$ and denote by x_4'', x_5'', x_6'' the resulting values of the variables: and hence

$$\frac{A_2''}{A_2} = \frac{x_4'}{x_4''} = e^{-\frac{x_3'}{x_5'}} = e^{-\frac{x_3 - x_1}{x_5 - x_1}}.$$

Thus

$$x_4'\delta x_3' + x_5'\delta x_4' + x_6'\delta x_5' = e^{-\frac{x_3 - x_1}{x_5 - x_1}} (x_5''\delta x_4'' + x_6''\delta x_5''),$$

and therefore

$$x_2\delta x_1 + x_3\delta x_2 + x_4\delta x_3 + x_5\delta x_4 + x_6\delta x_5 + x_1\delta x_6 = x_3'\delta x_2' + e^{-\frac{x_3 - x_1}{x_5 - x_1}} (x_5''\delta x_4'' + x_6''\delta x_5''),$$

which is evidently the final reduction of Ω. The second integral of $\Omega = 0$ is evidently

$$x_4'' = \text{constant} = c_2;$$

the equation then to be integrated is

$$x_6'' dx_5'' = 0,$$

the integral of which is

$$x_5'' = \text{constant} = c_3.$$

The three integrals of the original equation are thus

$$x_2' = c_1, \quad x_4'' = c_2, \quad x_5'' = c_3,$$

or, as is easily found by solving the equations which determine these variables,

$$\left.\begin{aligned} x_2' &= \frac{x_2x_3 + x_1x_4 + x_2x_5 + x_1x_6 - x_1x_2}{x_3 + x_5 - 2x_1} = c_1 \\ x_4'' &= \frac{x_1x_2 + x_1x_6 + x_4x_3 + x_4x_5 - x_1x_4}{x_3 + x_5 - 2x_1}\, e^{\frac{x_3 - x_1}{x_5 - x_1}} = c_2 \\ x_5'' &= x_5 - x_1 = c_3 \end{aligned}\right\};$$

and the coefficients which occur in the transformed value of Ω are

$$x_3' = x_3 - x_1,$$

$$\frac{x_6''}{x_4''} = \frac{x_6(x_3 + x_5) + x_1(x_4 + x_2) - x_1x_6}{x_1(x_2 + x_6) + x_4(x_3 + x_5) - x_1x_4} + \frac{x_3 - x_1}{x_5 - x_1}.$$

Ex. 2. Integrate

$$(x_2 + x_3)\,dx_1 + (x_3 + x_4)\,dx_2 + (x_4 + x_5)\,dx_3 + (x_5 + x_6)\,dx_4 + (x_6 + x_1)\,dx_5 + (x_1 + x_2)\,dx_6 = 0.$$

93. We now pass to the case in which the number of variables in equation (I) is originally odd; we have, by (2) of § 87,

$$\Omega = \sum_{m=1}^{2n+1} X_m \delta x_m = \lambda\delta\phi + \sum_{s=1}^{n} U_s \delta u_s \quad \ldots\ldots\ldots\ldots(2).$$

One of the integrals of $\Omega = 0$ is $\phi = a$; if, by means of the equations

$$\phi = a, \quad \delta\phi = \sum_{m=1}^{2n+1} \frac{\partial\phi}{\partial x_m} \delta x_m,$$

we eliminate from Ω the quantities x_{2n+1} and δx_{2n+1}, then Ω takes the form

$$\Omega = \sum_{m=1}^{2n+1} X_m \delta x_m = \sum_{s=1}^{2n} V_s \delta x_s + \lambda\delta\phi = \Theta + \lambda\delta\phi,$$

where

$$\lambda = X_{2n+1} \frac{1}{\dfrac{\partial\phi}{\partial x_{2n+1}}},$$

$$V_s = X_s - X_{2n+1} \frac{\dfrac{\partial\phi}{\partial x_s}}{\dfrac{\partial\phi}{\partial x_{2n+1}}}.$$

The expression Θ contains only $2n$ variables, and therefore the equation $\Theta = 0$ has, by the preceding investigation (§§ 88—92), a system of n integrals; and, if the principal integrals of the subsidiary equations be introduced, then

$$\Theta = \frac{A_1'}{A_1} V_2' \delta x_2' + \frac{A_1' A_2''}{A_1 A_2} V_4'' \delta x_4'' + \frac{A_1' A_2'' A_3'''}{A_1 A_2 A_3} V_6''' \delta x_6''' + \ldots\ldots,$$

where x_2', x_4'', x_6''', ... are principal integrals of subsidiary systems analogous to (6). The first of these systems would be obtained by replacing X in (6) by V and eliminating x_{2n+1} from V by means of $\phi = a$; and, after integration, a would be replaced by ϕ. .

We may however construct the subsidiary equations directly in the manner of § 89. If, in the equation

$$\sum_{m=1}^{2n+1} X_m \delta x_m - \lambda\delta\phi = \sum_{s=1}^{n} U_s \delta u_s,$$

we assume that the independent variable enters only as a factor

$\frac{1}{t_1}\left(=\frac{1}{A}\right)$ common to the coefficients on the right-hand side and we take $\mu = A\lambda$, then

$$\sum_{m=1}^{2n+1} AX_m \delta x_m = \mu\delta\phi + \sum_{s=1}^{n} a_s \delta u_s$$

with the earlier notation. The variables in the new equivalent form are $\phi, u_1, \ldots, u_n, t_1, \ldots, t_n$, among which there is no identical functional relation; and so we have

$$\sum_{m=1}^{2n+1} AX_m \frac{\partial x_m}{\partial t_r} = 0$$

(for $r = 1, 2, \ldots, n$), since the variations of δt do not occur on the right-hand side.

The last equation is an identity, when the proper values for x are substituted: taking any arbitrary variation in the result, we have, for $r = 1$,

$$\delta\left\{\sum_{m=1}^{2n+1} AX_m \frac{\partial x_m}{\partial t_1}\right\} = 0,$$

or

$$A \sum_{m=1}^{2n+1} \frac{\partial x_m}{\partial t_1} \delta X_m + A \sum_{m=1}^{2n+1} X_m \delta \frac{\partial x_m}{\partial t_1} + \delta A \sum_{m=1}^{2n+1} X_m \frac{\partial x_m}{\partial t_1} = 0,$$

so that, as in the last term the quantity multiplying δA vanishes, we have

$$A \sum_{m=1}^{2n+1} \frac{\partial x_m}{\partial t_1} \delta X_m + A \sum_{m=1}^{2n+1} X_m \delta \frac{\partial x_m}{\partial t_1} = 0.$$

But we have

$$\sum_{m=1}^{2n+1} AX_m \delta x_m - \mu\delta\phi$$

equal to $\sum_{s=1}^{n} a_s \delta u_s$ and therefore explicitly independent of t_1 (just as in the hypothesis of § 89 with the values there adopted for the coefficients a) when the proper substitutions are made for x: hence

$$\frac{\partial}{\partial t_1}\left\{\sum_{m=1}^{2n+1} AX_m \delta x_m - \mu\delta\phi\right\} = 0,$$

or, since ϕ is explicitly independent of t_1, we have

$$\begin{aligned}\frac{\partial \mu}{\partial t_1}\delta\phi &= \frac{\partial}{\partial t_1}\left\{\sum_{m=1}^{2n+1} AX_m \delta x_m\right\}\\ &= \frac{\partial A}{\partial t_1}\sum_{m=1}^{2n+1} X_m \delta x_m + A\sum_{m=1}^{2n+1} \frac{\partial X_m}{\partial t_1}\delta x_m + A\sum_{m=1}^{2n+1} X_m \frac{\partial}{\partial t_1}(\delta x_m).\end{aligned}$$

Now δx_m is an arbitrary variation, so that

$$\frac{\partial}{\partial t_1}(\delta x_m) = \delta \frac{\partial x_m}{\partial t_1},$$

and therefore, from the two equations which involve these equal quantities, we have

$$\frac{\partial \mu}{\partial t_1}\delta\phi = \frac{\partial A}{\partial t_1}\sum_{m=1}^{2n+1} X_m \delta x_m + A \sum_{m=1}^{2n+1}\left(\frac{\partial X_m}{\partial t_1}\delta x_m - \frac{\partial x_m}{\partial t_1}\delta X_m\right).$$

We have $\frac{\partial A}{\partial t_1} = 1$, because $A = t_1$; and

$$\frac{\partial X_m}{\partial t_1} = \sum_{s=1}^{2n+1}\frac{\partial X_m}{\partial x_s}\frac{\partial x_s}{\partial t_1},$$

$$\delta\phi = \sum_{s=1}^{2n+1}\frac{\partial \phi}{\partial x_s}\delta x_s, \qquad \delta X_m = \sum_{s=1}^{2n+1}\frac{\partial X_m}{\partial x_s}\delta x_s.$$

Substituting these and remembering that the variations of the variables are arbitrary, so that the coefficients of any the same variation on the two sides of the equation are equal, we have the $2n+1$ equations

$$\frac{\partial \mu}{\partial t_1}\frac{\partial \phi}{\partial x_s} = X_s + A\sum_{m=1}^{2n+1} a_{s,m}\frac{\partial x_m}{\partial t_1},$$

and therefore

$$X_s = \frac{\partial \phi}{\partial x_s}\frac{\partial \mu}{\partial t_1} + t_1 \sum_{m=1}^{2n+1} a_{m,s}\frac{\partial x_m}{\partial t_1} \quad \text{..................(8)},$$

an equation which holds for $s = 1, 2, \ldots, 2n+1$; and, since ϕ is independent of t_1 when the substitutions are made for x, we have

$$\sum_{m=1}^{2n+1}\frac{\partial \phi}{\partial x_m}\frac{\partial x_m}{\partial t_1} = 0 \quad \text{.....................(8}^{a}\text{)}.$$

These $2n+2$ equations are satisfied in the first instance as in the earlier case by

$$\phi = a,\ u_1,\ \ldots\ldots,\ u_n,\ t_2,\ t_3,\ \ldots\ldots,\ t_n,$$

(being $2n$ integrals) and by the subsequently deduced values of t_1 and of μ. If among the $2n+2$ equations we eliminate t_1 and μ, we have $2n$ equations remaining which involve $2n+1$ variables $x_1, x_2, \ldots, x_{2n+1}$; and these have the system of $2n$ integrals $\phi, u_1, \ldots, u_n, t_2, \ldots, t_n$.

94. We now, as in § 91, introduce the principal integrals of the subsidiary system (8). The integral $\phi = a$ is retained; we take $x_1 = 0$ and denote the remaining principal integrals by x_2', x_3', ..., x'_{2n}, the (unnecessary) value of x'_{2n+1} being then given by

$$\phi(x_1, x_2, \ldots, x_{2n+1}) = \phi(0, x_2', \ldots, x'_{2n+1}).$$

When these are substituted, we have

$$\sum_{m=1}^{2n+1} AX_m \,\delta x_m = A\left(\lambda\delta\phi + \sum_{s=1}^{2n} V_s \delta x_s\right)$$

$$= \mu\delta\phi + A' \sum_{s=2}^{2n} V_s' \,\delta x_s'.$$

If we take $\phi = a$, $x_2' = c_1$ (so that we have one determined, in addition to one arbitrary, integral) then the equation to be integrated is

$$\sum_{s=3}^{2n} V_s' \,dx_s' = 0$$

containing $2n - 2$ variables; or, if we merely take $x_2' = c_1$ as the single integral, then the equation to be integrated arises from the differential expression

$$\mu\delta\phi + A' \sum_{s=3}^{2n} V_s' \,\delta x_s',$$

which contains $2n - 1$ variables. In either case, the proper alternative of the two methods for an even or an odd number of variables respectively may be applied and the solution be thus gradually obtained; but the earlier of the two methods would in general prove the easier.

95. The subsidiary equations (8) evidently occur in a form different from that which characterised the subsidiary equations in the corresponding case of Pfaff's reduction, but the coefficients of the quantities $\dfrac{\partial x}{\partial t}$ on the right-hand side of (8) are the same as in the equations (I. c.) in that reduction. The determinant of those coefficients in the $2n+1$ equations (8) vanishes, being a skew determinant of odd order; but the earlier investigation (§ 65) shews that, if the equations be multiplied by the Pfaffians

$$[2, 3, \ldots, 2n+1],\ [3, 4, \ldots, 2n+1, 1],\ [4, 5, \ldots, 2n+1, 1, 2], \ldots$$

in order and be added together, the terms involving the quantities $\frac{\partial x}{\partial t}$ all disappear. Hence we have

$$\frac{\partial \mu}{\partial t_1} = \frac{\sum\limits_{s=1}^{2n+1} X_s [s+1, s+2, \ldots, s-2, s-1]}{\sum\limits_{s=1}^{2n+1} \frac{\partial \phi}{\partial x_s} [s+1, s+2, \ldots, s-2, s-1]} \quad \ldots\ldots\ldots\ldots(9),$$

so that, when t_1 is known, the integration of this equation at once gives the value of μ.

We may now use any $2n$ of the equations (8) together with (8^a) to determine the quantities $t_1 \frac{\partial x}{\partial t_1}$. Retaining the first $2n$ of them and writing

$$\Theta_s = X_s - \frac{\partial \mu}{\partial t_1} \frac{\partial \phi}{\partial x_s} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10),$$

where $\frac{\partial \mu}{\partial t_1}$ has the value given in (9), the equations are

$$\Theta_s + t_1 a_{s, 2n+1} \frac{\partial x_{2n+1}}{\partial t_1} = t_1 \sum_{m=1}^{2n} a_{m,s} \frac{\partial x_m}{\partial t_1},$$

for $s = 1, 2, \ldots, 2n$. Using now the solution as given in § 59 we have

$$(-1)^{m-1} [1, 2, 3, \ldots, 2n]\, t_1 \frac{\partial x_m}{\partial t_1} = W_m + t_1 \frac{\partial x_{2n+1}}{\partial t_1} \Delta_m,$$

where

$$W_m = \sum_{s=1}^{2n} \Theta_s [s+1,\ s+2,\ \ldots,\ s-1],$$

in which for every term under the sign of summation on the right-hand side the integers $s, s+1, \ldots, s-1$ are the integers $1, 2, \ldots, m-1, m+1, \ldots, 2n$ in their cyclical order, and

$$\begin{aligned}\Delta_m &= \sum_{s=1}^{2n} a_{s,2n+1} [s+1,\ s+2,\ \ldots,\ s-1] \\ &= [1,\ 2,\ 3,\ \ldots,\ 2n+1],\end{aligned}$$

in which m does not occur in the sequence $1, 2, \ldots, 2n+1$ of integers. But we also have, by (8^a),

$$\frac{\partial \phi}{\partial x_1} \frac{\partial x_1}{\partial t_1} + \frac{\partial \phi}{\partial x_2} \frac{\partial x_2}{\partial t_1} + \ldots\ldots + \frac{\partial \phi}{\partial x_{2n+1}} \frac{\partial x_{2n+1}}{\partial t_1} = 0;$$

whence, on multiplying by $[1, 2, 3, \ldots, 2n]\, t_1$, we have

$$t_1 \frac{\partial x_{2n+1}}{\partial t_1} \sum_{m=1}^{2n+1} \left\{(-1)^{m-1} \frac{\partial \phi}{\partial x_m} \Delta_m\right\} = \sum_{m=1}^{2n} (-1)^m \frac{\partial \phi}{\partial x_m} W_m.$$

Taking first the part on the right-hand side, the terms involving $\frac{\partial \mu}{\partial t_1}$, which occurs in Θ_s when we substitute for W_m, have, as the coefficient of $\frac{\partial \mu}{\partial t_1}$, the quantity

$$\sum_{m=1}^{2n} (-1)^{m-1} \frac{\partial \phi}{\partial x_m} \sum_{s=1}^{2n} \frac{\partial \phi}{\partial x_s} [s+1, s+2, \ldots, s-1],$$

the numbers $s+1, s+2, \ldots, s-1$ being the numbers $1, 2, \ldots, 2n$ with the omission of s and m, it being necessary that s and m are different integers. In this double summation the coefficient of $\frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j}$ is

$$\begin{aligned}(-1)^{i-1} [j+1, j+2, \ldots, j-1] &+ (-1)^{j-1} [i+1, i+2, \ldots, i-1] \\ &= \{(-1)^{i-1} + (-1)^{j-1+j-i-1}\} [j+1, j+2, \ldots, j-1] = 0.\end{aligned}$$

Hence the right-hand side of the equation giving $\frac{\partial x_{2n+1}}{\partial t_1}$ is

$$\begin{aligned}&\sum_{m=1}^{2n} (-1)^m \frac{\partial \phi}{\partial x_m} \sum_{s=1}^{2n} X_s [s+1, s+2, \ldots, s-1] \\ &= \sum_{m=1}^{2n} \sum_{s=1}^{2n} (-1)^m \frac{\partial \phi}{\partial x_m} X_s [s+1, s+2, \ldots, s-1],\end{aligned}$$

where in the symbol of the Pfaffian the integers $s, s+1, s+2, \ldots$ are the integers $1, 2, \ldots, 2n$ (with m omitted) taken in cyclical order. And the coefficient of $t_1 \frac{\partial x_{2n+1}}{\partial t_1}$ is

$$\begin{aligned}&\sum_{m=1}^{2n+1} (-1)^{m-1} \frac{\partial \phi}{\partial x_m} \Delta_m \\ &= \sum_{m=1}^{2n+1} \frac{\partial \phi}{\partial x_m} [m+1, m+2, \ldots, m-1],\end{aligned}$$

where the integers $m, m+1, m+2, \ldots$ are the integers $1, 2, \ldots, 2n+1$ taken in cyclical order. As this coefficient is symmetrical

in regard to all the variables, we may conveniently introduce a symbol for it: let

$$\nabla = \sum_{m=1}^{2n+1} \frac{\partial \phi}{\partial x_m} [m+1, m+2, \ldots, m-1] \quad \ldots\ldots(11);$$

it will be noticed that ∇ is the denominator of $\frac{\partial \mu}{\partial t_1}$ in (9). Then

$$\nabla t_1 \frac{\partial x_{2n+1}}{\partial t_1} = \sum_{m=1}^{2n} \sum_{s=1}^{2n} (-1)^m \frac{\partial \phi}{\partial x_m} X_s [s+1, s+2, \ldots, s-1] \ldots(12').$$

96. This value may be substituted in the equations, which express $\frac{\partial x_m}{\partial t_1}$ in terms of $\frac{\partial x_{2n+1}}{\partial t_1}$, to find the values of the quantities $\frac{\partial x_m}{\partial t_1}$; or the equations may be solved again, taking $\frac{\partial x_m}{\partial t_1}$ as the initially undetermined quantity, in place of $\frac{\partial x_{2n+1}}{\partial t_1}$. In either case, *the result which includes the values of all these coefficients may be expressed in the form*

$$(-1)^{p-1} \nabla t_1 \frac{\partial x_p}{\partial t_1} = \sum_{m=1}^{2n+1} \sum_{s=1}^{2n+1} (-1)^{m'} \frac{\partial \phi}{\partial x_m} X_s [s+1, s+2, \ldots, s-1] \quad (12),$$

where $s+1, s+2, \ldots, s-1$ *are the integers* $1, 2, \ldots, 2n+1$ (*with* s, m, p *omitted*) *taken in cyclical order beginning with* $s+1$; *the integers* s, m, p *are to be different from one another, so that the values* $m=p$, $s=p$ *may not occur on the right-hand side and the terms corresponding to* $s=m$ *do not occur; and the value of* m' *is* m, *when* $m<p$, *and is* $m-1$, *when* $m>p$.

In particular, when there are three variables (so that $n=1$) we have

$$\left.\begin{aligned}
\nabla &= [23]\frac{\partial \phi}{\partial x_1} + [31]\frac{\partial \phi}{\partial x_2} + [12]\frac{\partial \phi}{\partial x_3} \\
\nabla \frac{\partial \mu}{\partial t_1} &= [23]X_1 + [31]X_2 + [12]X_3 \\
\nabla t_1 \frac{\partial x_1}{\partial t_1} &= \qquad\qquad X_2 \frac{\partial \phi}{\partial x_3} - X_3 \frac{\partial \phi}{\partial x_2} \\
-\nabla t_1 \frac{\partial x_2}{\partial t_1} &= X_1 \frac{\partial \phi}{\partial x_3} \qquad\qquad - X_3 \frac{\partial \phi}{\partial x_1} \\
\nabla t_1 \frac{\partial x_3}{\partial t_1} &= X_1 \frac{\partial \phi}{\partial x_2} - X_2 \frac{\partial \phi}{\partial x_1}
\end{aligned}\right\};$$

and when there are five variables (so that $n = 2$) we have

$$\left.\begin{array}{rl}
\nabla = & [2345]\dfrac{\partial\phi}{\partial x_1} + [3451]\dfrac{\partial\phi}{\partial x_2} + [4512]\dfrac{\partial\phi}{\partial x_3} + [5123]\dfrac{\partial\phi}{\partial x_4} + [1234]\dfrac{\partial\phi}{\partial x_5} \\
\nabla\dfrac{\partial\mu}{\partial t_1} = & [2345]X_1 + [3451]X_2 + [4512]X_3 + [5123]X_4 + [1234]X_5 \\
\nabla t_1\dfrac{\partial x_1}{\partial t_1} = & \qquad\qquad - \{345\}\dfrac{\partial\phi}{\partial x_2} + \{245\}\dfrac{\partial\phi}{\partial x_3} - \{235\}\dfrac{\partial\phi}{\partial x_4} + \{234\}\dfrac{\partial\phi}{\partial x_5} \\
-\nabla t_1\dfrac{\partial x_2}{\partial t_1} = & -\{345\}\dfrac{\partial\phi}{\partial x_1} \qquad\qquad + \{145\}\dfrac{\partial\phi}{\partial x_3} - \{135\}\dfrac{\partial\phi}{\partial x_4} + \{134\}\dfrac{\partial\phi}{\partial x_5} \\
\nabla t_1\dfrac{\partial x_3}{\partial t_1} = & -\{245\}\dfrac{\partial\phi}{\partial x_1} + \{145\}\dfrac{\partial\phi}{\partial x_2} \qquad\qquad - \{125\}\dfrac{\partial\phi}{\partial x_4} + \{124\}\dfrac{\partial\phi}{\partial x_5} \\
-\nabla t_1\dfrac{\partial x_4}{\partial t_1} = & -\{235\}\dfrac{\partial\phi}{\partial x_1} + \{135\}\dfrac{\partial\phi}{\partial x_2} - \{125\}\dfrac{\partial\phi}{\partial x_3} \qquad\qquad + \{123\}\dfrac{\partial\phi}{\partial x_5} \\
\nabla t_1\dfrac{\partial x_5}{\partial t_1} = & -\{234\}\dfrac{\partial\phi}{\partial x_1} + \{134\}\dfrac{\partial\phi}{\partial x_2} - \{124\}\dfrac{\partial\phi}{\partial x_3} + \{123\}\dfrac{\partial\phi}{\partial x_4}
\end{array}\right\},$$

in which the symbol $\{\lambda, \mu, \nu\}$ is defined by

$$\{\lambda, \mu, \nu\} = \{\mu, \nu, \lambda\} = \{\nu, \lambda, \mu\} = X_\lambda[\mu\nu] + X_\mu[\nu\lambda] + X_\nu[\lambda\mu].$$

97. From the value of $\dfrac{\partial\mu}{\partial t_1}$ as given in (9) we have an immediate verification of Jacobi's result given in § 65. If

$$\sum_{s=1}^{2n+1} X_s[s+1, s+2, \ldots, s-1] = 0,$$

then $\dfrac{\partial\mu}{\partial t_1}$ vanishes and therefore μ when expressed in terms of the new variables is explicitly independent of t_1; and in that case the differential equation to be integrated is, when transformed,

$$\mu\, d\phi + \sum_{s=1}^{n} \alpha_s\, du_s = 0,$$

in which the variables are $\phi, u_1, \ldots, u_n, t_2, \ldots, t_n$, which are $2n$ in number. But such an equation, involving only $2n$ variables, has its integral equivalent constituted by n equations—in the present case after only a single (§ 92) integration of subsidiary equations; and hence we infer the result already obtained (§ 66), viz. that the equation

$$\sum_{m=1}^{2n+1} X_m\, dx_m = 0$$

can be represented by n integral equations, when the condition

$$\sum_{s=1}^{2n+1} X_s [s+1, s+2, \ldots, s-1] = 0$$

is identically satisfied.

98. Natani does not give the solutions of the equations (12) which have just been obtained; it appears evident from their not too simple form that the derivation of the integrals and thence of the principal integrals would, even for a specified unarbitrary function ϕ, be a matter of some difficulty. In actual practice probably the simplest method would be to reduce the equation

$$\Omega = \sum_{m=1}^{2n+1} X_m dx_m = 0$$

by means of the arbitrarily assumed integral $\phi = a$ to an equation $\Omega' = 0$ free from x_{2n+1} and dx_{2n+1}, that is, to an equation involving an even number of variables; and to apply to this reduced equation the appropriate earlier method, as given in §§ 59, 60.

Thus by taking the very special form

$$\phi = x_{2n+1}$$

we may derive the former system (§ 59) of subsidiary equations: for $\dfrac{\partial \phi}{\partial x_m}$ vanishes for $m = 1, 2, \ldots, 2n$ and we also have

$$\nabla = [1, 2, \ldots, 2n],$$

which is the modified form of (11), and

$$(-1)^{p-1} \nabla t_1 \frac{\partial x_p}{\partial t_1} = \sum_{s=1}^{2n} X_s [s+1, s+2, \ldots, s-1],$$

where $s+1, s+2, \ldots, s-1$ are the integers $1, 2, \ldots, 2n$ (with s and p omitted) taken in cyclical order.

99. Hitherto it has been assumed (§ 89) that no relations exist among the coefficients X in the differential equation, and the consequent minimum number of equations in the integral system has been indicated. It may however happen that some relations are satisfied which will reduce this number to be less than the general minimum: to the consideration of these relations we now proceed.

100. We take first the case of an *even* number of variables in the original differential equation.

The subsidiary equations (6) are, when the variable t_1 is eliminated, $2n-1$ in number; and in the general case their integrals are the quantities $u_1, u_2, \ldots, u_n$ and the quantities $t_2, t_3, \ldots, t_n$ or (what is equivalent to the latter set) $U_2/U_1, U_3/U_1, \ldots$ $\ldots, U_n/U_1$, being a system of $2n-1$ integrals, the proper number. But if the original differential equation can be reduced to the form

$$U_1du_1 + U_2du_2 + \ldots\ldots + U_qdu_q = 0,$$

then there will be only $2q-1$ integrals viz., $u_1, \ldots, u_q, U_2/U_1, \ldots,$ U_q/U_1. Since the $2n-1$ equations have only $2q-1$ independent integrals, it follows that $2n-2q$ of the equations must be derivable from the remaining $2q-1$ equations; the conditions for this derivation, being the conditions that there are only q differential elements in the transformed expression for Ω, will be the conditions that $\Omega = 0$ can be satisfied by $q\,(< n)$ integrals.

Since $2n-2q$ of the $2n$ equations (§ 89)

$$X_s = t_1 \sum_{m=1}^{2n} a_{m,s} \frac{\partial x_m}{\partial t_1}$$

are derivable linearly from the remainder, we must have, on the supposition that the first $2q$ of the equations are independent, relations of the form

$$X_r = \sum_{t=1}^{2q} c_{r,t}\, X_t,$$

$$a_{p,r} = \sum_{t=1}^{2q} c_{r,t}\, a_{p,t},$$

where r has the values $2q+1, 2q+2, \ldots, 2n$ and p has the values $1, 2, \ldots, 2n$; and the quantities $c_{r,t}$, which are of the nature of indeterminate multipliers, must be eliminated before the conditions can be obtained.

When these quantities c are eliminated, the resulting equations are of two kinds. There are equations of the form

$$\begin{vmatrix} 0, & a_{12}, & a_{13}, & \ldots\ldots, & a_{1,2q}, & a_{1,r} \\ a_{21}, & 0, & a_{23}, & \ldots\ldots, & a_{2,2q}, & a_{2,r} \\ a_{31}, & a_{32}, & 0, & \ldots\ldots, & a_{3,2q}, & a_{3,r} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{2q,1}, & a_{2q,2}, & a_{2q,3}, & \ldots\ldots, & 0, & a_{2q,r} \\ X_1, & X_2, & X_3, & \ldots\ldots, & X_{2q}, & X_r \end{vmatrix} = 0 \ldots (13),$$

where r has the values $2q+1,\ 2q+2,\ \ldots,\ 2n$; and there are equations of the form

$$\begin{vmatrix} 0\ , & a_{12}\ , & a_{13}\ , & \ldots\ldots, & a_{1,2q}, & a_{1,r} \\ a_{21}\ , & 0\ , & a_{23}\ , & \ldots\ldots, & a_{2,2q}, & a_{2,r} \\ a_{31}\ , & a_{32}\ , & 0\ , & \ldots\ldots, & a_{3,2q}, & a_{3,r} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{2q,1}, & a_{2q,2}, & a_{2q,3}, & \ldots\ldots, & 0\ , & a_{2q,r} \\ a_{s,1}\ , & a_{s,2}, & a_{s,3}, & \ldots\ldots, & a_{s,2q}, & a_{s,r} \end{vmatrix} = 0 \ldots (14),$$

where the possible values of r are $2q+1,\ 2q+2,\ \ldots,\ 2n$ and those of s are $2q+1, 2q+2,\ \ldots,\ 2n$.

The former set (13) are independent of one another, and so contribute $2n-2q$ conditions.

The latter set (14) furnish conditions only if r and s be different integers, for when $r=s$ the resulting equation (arising from a skew determinant of odd order) is evanescent. Also the same condition is furnished by $r=i,\ s=j$ as by $r=j,\ s=i$, on account of the relation

$$a_{k,l}=-a_{l,k};$$

hence the number of conditions is the number of pairs of different integers from the series $2q+1,\ 2q+2, \ldots,\ 2n$, that is, it is

$$\tfrac{1}{2}(2n-2q)(2n-2q-1).$$

Hence *the total number of independent conditions necessary that the equation*

$$\sum_{m=1}^{2n} X_m dx_m = 0$$

may be satisfied by $q\ (<n)$ *integral equations is*

$$2n-2q+\tfrac{1}{2}(2n-2q)(2n-2q-1)=(n-q)(2n-2q+1);$$

and the conditions are the equations (13) *for the values* $r=2q+1$, $2q+2, \ldots,\ 2n$ *and the equations* (14) *for all possible pairs of different integers from the series* $2q+1,\ 2q+2, \ldots,\ 2n$ *for* r *and* s.

In particular, if $q=1$ so that the equation is exact, the number of independent conditions is $(n-1)(2n-1)$, which agrees with the number previously (§ 6) obtained.

101. We now take the case of an *odd* number of variables in the original differential equation.

If the arbitrary integral exist, it evidently from (12) enters into the subsidiary equations; and then the simpler plan will be to use the arbitrary integral to eliminate some one of the variables and the differential element of that variable and thus to reduce the equation to one which contains the next lower even number of variables. The conditions that the new equation may be satisfied by a number of integral equations smaller than the general minimum (which will occur if the old equation be so satisfied) may be obtained from the preceding investigation.

If the arbitrary integral do not exist*, then the subsidiary equations (8) take the form

$$X_s = t_1 \sum_{m=1}^{2n+1} a_{m,s} \frac{\partial x_m}{\partial t_1} \quad \text{.....................(8')},$$

and they are $2n+1$ in number; when t_1 is eliminated, there are $2n$ equations. This number must however be reduced by unity on account of the relation

$$\sum_{s=1}^{2n+1} X_s [s+1, s+2, \dots, s-1] = 0 \quad \text{...........(15)},$$

which is the necessary condition that the modified equations (8′) may coexist.

Supposing then that the given differential equation can be satisfied by q integral equations, we have $2q-1$ integrals of the foregoing subsidiary equations in the forms $u_1, u_2, \dots, u_q, U_2/U_1, \dots, U_q/U_1$. Now, by the satisfaction of the foregoing single condition, the last of the equations (8′) can be derived from the first $2n$ of them and therefore may for the present be omitted from consideration. The first $2q$ of the equations (8′) will suffice to determine t_1 and the desired $2q-1$ integrals; and therefore the remaining $2n-2q$ equations in (8′) must be derivable from the first $2q$ of them. The conditions of this derivation are the conditions that the differential equation may be satisfied by q integrals.

* Natani gives no means of determining whether an arbitrary integral does or does not exist. Subsequent investigations (of Clebsch, and especially of Lie) render this determination unnecessary, owing to the modifications effected in the reduced form so as to make it normal.

The analysis is similar to that in the preceding case, and the result is that there are $2n-2q$ equations of the form

$$\begin{vmatrix} 0 & , a_{12} & , a_{13} & ,\ldots\ldots, & a_{1,2q}, & a_{1,r} \\ a_{21} & , 0 & , a_{23} & ,\ldots\ldots, & a_{2,2q}, & a_{2,r} \\ a_{31} & , a_{32} & , 0 & ,\ldots\ldots, & a_{3,2q}, & a_{3,r} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{2q,1}, & a_{2q,2}, & a_{2q,3}, & \ldots\ldots, & 0 & , a_{2q,r} \\ X_1 & , X_2 & , X_3 & ,\ldots\ldots, & X_{2q}, & X_r \end{vmatrix} = 0 \quad \ldots\ldots(16),$$

for $r = 2q+1, 2q+2, \ldots, 2n$; and there are equations of the form

$$\begin{vmatrix} 0 & , a_{12} & , a_{13} & ,\ldots\ldots, & a_{1,2q}, & a_{1,r} \\ a_{21} & , 0 & , a_{23} & ,\ldots\ldots, & a_{2,2q}, & a_{2,r} \\ a_{31} & , a_{32} & , 0 & ,\ldots\ldots, & a_{3,2q}, & a_{3,r} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{2q,1}, & a_{2q,2}, & a_{2q,3}, & \ldots\ldots, & 0 & , a_{2q,r} \\ a_{s,1} & , a_{s,2} & , a_{s,3} & ,\ldots\ldots, & a_{s,2q}, & a_{s,r} \end{vmatrix} = 0 \quad \ldots\ldots(17),$$

for the values $2q+1$, $2q+2, \ldots, 2n$ of r and the values $2q+1$, $2q+2, \ldots, 2n+1$ of s. The number of equations in (17) is:

$$2n-2q, \text{ for } s = 2n+1 \text{ and } r = 2q+1,\ 2q+2, \ldots, 2n;$$

together with

$$\tfrac{1}{2}(2n-2q)(2n-2q-1)$$

for the different pair-combinations of r and s from the series

$$2q+1,\ 2q+2, \ldots, 2n:$$

hence the total number of these equations (17) is

$$\tfrac{1}{2}(2n-2q)(2n-2q+1).$$

And, lastly, there is the condition (15).

Hence it follows that, *if the single condition* (15), *the* $2n-2q$ *conditions* (16) *and the* $(n-q)(2n-2q+1)$ *conditions* (17)—*making in all* $(2n-2q+1)(n-q+1)$ *conditions—be satisfied, then the equation*

$$\sum_{m=1}^{2n+1} X_m dx_m = 0$$

can be satisfied by q integral determinate equations.

In particular, if $q = 1$ so that the equation is exact, the number of independent conditions is $n(2n-1)$, which agrees with the number previously (§ 6) obtained.

102. The integration of the subsidiary equations must now be considered. Suppose that the original differential equation is, owing to the satisfaction of the necessary conditions, satisfied by a system of q integral equations; then the preceding investigations shew that $2q - 1$ quantities (independent of t_1) must be determined and therefore that the number of subsidiary equations independent of one another and free from t_1 is $2q - 1$. These quantities involve all the variables of the original differential equation; and therefore in the subsidiary equations we treat $2q - 1$ of the variables as dependent and the remainder—viz., $2n - 2q + 1$ or $2n - 2q + 2$ according as the original number is even or odd—as independent. Let then s denote this number—$2n - 2q + 1$ or $2n - 2q + 2$ in the two cases—and let the independent variables be taken to be $x_1, x_2, \dots, x_s$; let the dependent variables be taken to be $y_1, y_2, \dots, y_{2q-1}$; then the $2q - 1$ subsidiary equations, when t_1 is eliminated, are all of the form

$$\sum_{r=1}^{s} Z_{r,\mu}\, dx_r + \sum_{i=1}^{2q-1} T_{i,\mu}\, dy_i = 0,$$

for $\mu = 1, 2, \dots, 2q - 1$; and the coefficients Z and T are of the form

$$X_{2q} a_{m,\mu} - X_\mu a_{m,2q}.$$

This is a system of $2q - 1$ differential equations in more than $2q$ variables and it is satisfied by a system of $2q - 1$ integral equations. Hence it is a system of exact equations; and we obtain the integrals by one of the methods already indicated in Chap. II. If in particular we use Natani's process (l. c. § 33) and introduce the principal integrals, which will be $y_1^{(s)}, y_2^{(s)}, \dots, y_{2q-1}^{(s)}$ when after s integrations we make $x_1 = 0, x_2 = 0, \dots, x_s = 0$ as usual in that method, we may take $y_i^{(s)}$ to be $x_{s+i}^{(s)}$ and then the differential equation is

$$\sum_{m=1} X_m\, dx_m = \frac{A^{(s)}}{A} \sum_{i=1}^{2q-1} X_{s+i}^{(s)}\, dx_{s+i}^{(s)} = 0,$$

where $X^{(s)}_{s+i}$ is the value of X_{s+i} when we take $x_j=0$, for $j=1, \ldots, s$ and $x_{s+i}=x^{(s)}_{s+i}$, for $i=1, \ldots, 2q-1$.

We now have a differential equation

$$\sum_{i=1}^{2q-1} X^{(s)}_{s+i}\, dx^{(s)}_{s+i}=0$$

involving $2q-1$ variables and satisfied by q integrals; it is thus of the normal unconditioned form previously (§ 93) considered. We may take as a first integral

$$y^{(s)}_1=x^{(s)}_{s+1}=\text{constant};$$

and then the equation

$$\sum_{i=2}^{2q-1} X^{(s)}_{s+i}\, dx^{(s)}_{s+i}=0$$

in $2q-2$ variables has $q-1$ integrals which, with the one already obtained, form the system of q integral equations; or we may take as a first integral

$$\phi(x^{(s)}_{s+1}, \ldots)=\text{constant},$$

and, eliminating by this integral the quantities $x^{(s)}_{s+1}$ and $dx^{(s)}_{s+1}$ from the equation, we shall have a similar equation in $2q-2$ variables, to be integrated as before.

The second of these suppositions contemplates the reduced equation as one in an odd number of variables,—and the integral adopted is the usual necessary integral (§ 69) of arbitrary form which is taken for the integral of the equation. The first of them contemplates the equation as being the first reduced form in an equation which involves $2q$ variables; and the integral adopted is the usual (l.c.) first integral of such an equation. The relation between the two integral systems will be seen when we come to consider Clebsch's method.

Ex. 1. To integrate

$$(x_2-x_3+x_4-x_5)\,dx_1+(x_3-x_4+x_5-x_1)\,dx_2+(x_4-x_5+x_1-x_2)\,dx_3$$
$$+(x_5-x_1+x_2-x_3)\,dx_4+(x_1-x_2+x_3-x_4)\,dx_5=0.$$

For this equation we have

$$a_{12}=a_{23}=a_{34}=a_{45}=a_{14}=a_{25}=a_{31}=a_{42}=a_{53}=a_{51}=2,$$
$$[1234]=[2345]=[3451]=[4512]=[5123]=4;$$

the condition (15) is satisfied, but for $q=1$ none of the conditions (16) or (17) are satisfied and for $q=2$ the conditions (17) do not exist, while the only form surviving from (16) is (15). Hence the given equation can be satisfied by *two* integrals.

The subsidiary equations are

$$X_1=\frac{2t_1}{dt_1}(-dx_2+dx_3-dx_4+dx_5)=-2t_1\frac{\partial X_1}{\partial t_1},$$

$$X_2=-2t_1\frac{\partial X_2}{\partial t_1},\qquad X_3=-2t_1\frac{\partial X_3}{\partial t_1},$$

$$X_4=-2t_1\frac{\partial X_4}{\partial t_1},\qquad X_5=-2t_1\frac{\partial X_5}{\partial t_1},$$

of which only four are independent; they may be written

$$\frac{dX_1}{X_1}=\frac{dX_2}{X_2}=\frac{dX_3}{X_3}=\frac{dX_4}{X_4}\left[=\frac{dX_5}{X_5}=\frac{dt_1}{-2t_1}\right].$$

Three independent integrals are

$$u=\frac{X_2}{X_1},\qquad v=\frac{X_3}{X_1},\qquad w=\frac{X_4}{X_1};$$

from which we find

$$\begin{aligned}
x_2&=X_1(1+u)+x_1,\\
x_3&=X_1(1+2u+v)+x_1,\\
x_4&=X_1(2u+2v+w+1)+x_1,\\
x_5&=X_1(u+v+w)+x_1.
\end{aligned}$$

When these are substituted in $X_1dx_1+X_2dx_2+X_3dx_3+X_4dx_4+X_5dx_5$, it takes the form

$$X_1^2\{(v+w-1)\,du+(w-u-1)\,dv-(u+v+1)\,dw\},$$

thus giving an indirect verification of X_1^{-2} as the value of t_1 derived by the integration of the subsidiary equations: for $t_1\Omega$ is, in the new form, independent of t_1. Thus the equation to be integrated is

$$(v+w-1)\,du+(w-u-1)\,dv-(u+v+1)\,dw=0,$$

and, as is to be expected, it does not satisfy the condition of integrability. Its two integrals may be taken in the form

$$v=\text{constant},$$

$$\frac{u+v+1}{v+w-1}=\text{constant};$$

or two integrals, in virtue of which the original equation is satisfied, are

$$\frac{x_1-x_2+x_4-x_5}{x_2-x_3+x_4-x_5}=\text{constant},$$

$$\frac{x_4-x_5}{x_5-x_2}=\text{constant}.$$

Ex. 2. We may add, as a note on the foregoing, the remark that when the condition (15) and no other condition is satisfied for an equation in an odd number of variables (§ 65), then, without regard to the existence of the arbitrary integral in § 100, a reduction to the next smaller number of variables can often be made by omitting from the subsidiary equations the variation of some one variable.

Thus, if it be supposed that x_5 does not vary in the subsidiary equations (or what is the same thing if we find variables suitable for the transformation $\sum_{r=1}^{4} X_r dx_r$), these equations are

$$\frac{dx_1}{x_1-x_5}=\frac{dx_2}{x_2-x_5}=\frac{dx_3}{x_3-x_5}=\frac{dx_4}{x_4-x_5};$$

hence variables suitable to transformation are

$$u'=\frac{x_2-x_5}{x_1-x_5},\qquad v'=\frac{x_3-x_5}{x_1-x_5},\qquad w'=\frac{x_4-x_5}{x_1-x_5}.$$

Then we have

$$\begin{aligned} x_1&=\lambda \quad +x_5,\\ x_2&=\lambda u'+x_5,\\ x_3&=\lambda v'+x_5,\\ x_4&=\lambda w'+x_5; \end{aligned}$$

when these are substituted we find

$$\sum_{r=1}^{5} X_r dx_r=\lambda^2\{(v'-w'-1)\,du'+(w'-u'+1)\,dv'+(u'-v'-1)\,dw'\}=0,$$

so that the equation to be integrated is

$$(v'-w'-1)\,du'+(w'-u'+1)\,dv'+(u'-v'-1)\,dw'=0.$$

The integrals of this equation are

$$\frac{1-u'+w'}{u'-v'+w'}=\text{constant},$$

$$\frac{w'}{u'}=\text{constant}.$$

The general justification of the method is given in § 66.

103. The knowledge of one of the integrals of the subsidiary equations (and therefore, as it is the first, an integral of the given differential equation, supposed to be an equation in an even number of variables) can be used* as follows to diminish the number of equations in the subsidiary system.

* Natani's investigation was published before Jacobi's memoir in *Crelle*, t. LX.; otherwise the following results might have been replaced by results in that memoir. Natani's result here given coincides, though not in explicit expression, with the results obtained by Clebsch (§§ 121, 122).

Let u_1 be the integral which is known and $u_2, \ldots, u_n$ those which have still to be found. Then we have

$$\sum_{m=1}^{2n} X_m \delta x_m = \frac{1}{A} \sum_{s=1}^{n} \alpha_s \delta u_s,$$

where $A (= t_1)$ is the variable which occurs only as a common factor in coefficients in the right-hand side; the quantities α are independent of t_1, and we leave them in their general form not adopting the special forms of § 89. From this equation it follows that

$$\sum_{m=1}^{2n} A X_m \delta x_m - \alpha_1 \delta u_1 = \alpha_2 \delta u_2 + \ldots\ldots + \alpha_n \delta u_n \ldots\ldots\ldots (a).$$

Applying to this equation the same process as in § 93 and now bearing in mind that α_1 and t_1 are independent of one another so that there are two independent variables, we arrive at the $2n$ equations

$$X_s \delta t_1 = \frac{\partial u_1}{\partial x_s} \delta \alpha_1 + t_1 \sum_{m=1}^{2n} a_{m,s} \delta x_m \ldots\ldots\ldots\ldots\ldots (18),$$

for $s = 1, \ldots, 2n$.

Now two of these equations will be required to determine the new independent variables t_1 and α_1 as functions of the old variables x; and there will therefore be $2n - 2$ equations left to determine such quantities as remain. Now for the equation (a) there are only $2n - 3$ quantities to be determined from the equations (18), viz., $u_2, \ldots, u_n, \frac{\alpha_3}{\alpha_2}, \ldots, \frac{\alpha_n}{\alpha_2}$: since therefore the $2n - 2$ equations which survive from (18) have only $2n - 3$ integrals (being the foregoing quantities), one among those equations must be a merely linear combination of the other $2n - 3$.

Each of these equations involves two independent variables in their variations and therefore leads to two equations of the forms

$$X_s = t_1 \sum_{m=1}^{2n} a_{m,s} \frac{\partial x_m}{\partial t_1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{i}),$$

$$0 = \frac{\partial u_1}{\partial x_s} + t_1 \sum_{m=1}^{2n} a_{m,s} \frac{\partial x_m}{\partial \alpha_1} \ldots\ldots\ldots\ldots\ldots\ldots (\text{ii});$$

so that *we have two systems each containing $2n - 3$ independent equations*. The quantities t_1 and α_1 are determinable by quadra-

tures and u_1 is given; so that of the variables t_1 and the $2n$ variables x three may be considered as removed, and *the* $2n-3$ *equations in each system will contain* $2n-2$ *variables.*

Now when no integrals are known (i) is the subsidiary system; and it has $2n-1$ integrals, all independent of t_1. Of these $2n-1$ one is necessarily u_1; other $2n-3$ are those common to the double system (i) and (ii); and the remaining integral is evidently α_1.

If we take the solutions of the equations (i) in the form

$$t_1 \frac{\partial x_m}{\partial t_1} = \sum_{s=1}^{2n} X_s R_{m,s},$$

then the solutions of the equations (ii) are

$$t_1 \frac{\partial x_m}{\partial \alpha_1} = -\sum_{s=1}^{2n} \frac{\partial u_1}{\partial x_s} R_{m,s}.$$

Hence, if θ be any function of the variables x and it be expressed in terms of the new variables, we have

$$\begin{aligned}\frac{\partial \theta}{\partial t_1} &= \sum_{m=1}^{2n} \frac{\partial \theta}{\partial x_m} \frac{\partial x_m}{\partial t_1}\\ &= \frac{1}{t_1} \sum_{m=1}^{2n} \sum_{s=1}^{2n} X_s R_{m,s} \frac{\partial \theta}{\partial x_m};\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \theta}{\partial \alpha_1} &= \sum_{m=1}^{2n} \frac{\partial \theta}{\partial x_m} \frac{\partial x_m}{\partial \alpha_1}\\ &= -\frac{1}{t_1} \sum_{m=1}^{2n} \sum_{s=1}^{2n} \frac{\partial \theta}{\partial x_m} \frac{\partial u_1}{\partial x_s} R_{m,s}.\end{aligned}$$

It at once follows that, if θ be an integral of the Pfaffian differential equation, say an integral u_2 second to u_1, so that in the new variables it is independent of t_1 and of α_1, then it satisfies the two equations

$$\sum_{m=1}^{2n} \sum_{s=1}^{2n} X_s R_{m,s} \frac{\partial u_2}{\partial x_m} = 0,$$

$$\sum_{m=1}^{2n} \sum_{s=1}^{2n} \frac{\partial u_1}{\partial x_s} R_{m,s} \frac{\partial u_2}{\partial x_m} = 0$$

which agree with those given subsequently in Clebsch's theory.

104. Such being the effect of the knowledge of one of the integrals of the subsidiary equations (6) of § 89 or (i) above, which is also by the nature of the question an integral of the differential equation, let us now consider the *effect of knowing two integrals of the subsidiary equations* (6) *or* (i).

One of the two is an integral of the differential equation and so may be denoted by u_1; let the other be denoted by θ. Then, since both of them are integrals of (i), we have

$$\sum_{m=1}^{2n}\sum_{s=1}^{2n} X_s R_{m,s}\frac{\partial u_1}{\partial x_m}=0,$$

$$\sum_{m=1}^{2n}\sum_{s=1}^{2n} X_s R_{m,s}\frac{\partial \theta}{\partial x_m}=0.$$

Now θ may or may not be an integral of the differential equation; we have, by the last paragraph,

$$-t_1\frac{\partial\theta}{\partial\alpha_1}=\sum_{m=1}^{2n}\sum_{s=1}^{2n}\frac{\partial\theta}{\partial x_m}\frac{\partial u_1}{\partial x_s}R_{m,s};$$

and since θ is an integral of the system (i), it is a function of u_1, of α_1, and of the $2n-3$ solutions common to the systems (i) and (ii) or, say, of α_1 and these $2n-3$ solutions alone, for wherever u_1 occurs it may be replaced by a constant. Hence, whatever be the form of θ, we shall have $\dfrac{\partial\theta}{\partial\alpha_1}$ a function of these same $2n-3$ solutions and of α_1, so that $\dfrac{\partial\theta}{\partial\alpha_1}$ is also a solution of the system (i) or of the equivalent partial equation

$$\sum_{m=1}^{2n}\sum_{s=1}^{2n} X_s R_{m,s}\frac{\partial u}{\partial x_m}=0.$$

Four cases may occur, as to the above value of $\dfrac{\partial\theta}{\partial\alpha_1}$.

First case, $\dfrac{\partial\theta}{\partial\alpha_1}$ *may vanish.* This is, by the last paragraph, the condition that θ satisfies the equations which determine a second integral of the Pfaffian equation; and therefore in this case *θ is a second integral, say u_2.*

Second case, $\dfrac{\partial\theta}{\partial\alpha_1}$ *may be a pure constant,* say $\dfrac{1}{c}$; then we have $\alpha_1=c\theta$. Hence in this case *the second integral of the subsidiary*

system is a mere constant multiple of the coefficient of the differential element du_1 of the first integral in the formation of the reduced Pfaffian expression. This is slightly more advantageous than being compelled to determine α_1 by a quadrature.

Third case, $\dfrac{\partial\theta}{\partial\alpha_1}$ *may be a function of θ*, say $f(\theta)$; then we have

$$d\alpha_1 = \frac{d\theta}{f(\theta)}$$

or

$$\alpha_1 = \int \frac{d\theta}{f(\theta)},$$

so that α_1 is determined by a quadrature. Unless this quadrature be easier than that which has determined α_1, there is in the present case no advantage to be derived in this direction from the knowledge of the second integral of the subsidiary system.

Fourth case, $\dfrac{\partial\theta}{\partial\alpha_1}$ *may be a non-vanishing function of variables* which is not expressible in terms of θ; let it be denoted by v. Then v is a new solution of the subsidiary system (i); and it therefore *has the same possibilities as θ*. Hence v may be a second integral of the original Pfaffian differential equation; or, if $\dfrac{\partial v}{\partial\alpha_1}$ be either a constant or a function of θ and v, a very simple quadrature for the constant or a comparatively simple one for the functional value will determine α_1; or $\dfrac{\partial v}{\partial\alpha_1}$ may be different from all of these and so, as in the case of $\dfrac{\partial\theta}{\partial\alpha_1}$, be again a new solution of the subsidiary system (i).

As that subsidiary system has only a finite number of solutions, there will be a limit to the development of the successive possibilities represented by the fourth case: so that we shall ultimately obtain either a second integral of the Pfaffian equation or shall derive α_1 by quadratures.

105. Just as, in § 103, we investigated the modification of the subsidiary system rendered possible by the knowledge of one integral of the differential equation, so similarly we may investigate

the modifications rendered possible by the knowledge of *two integrals* of the differential equation.

Taking then two integrals, supposed known, to be u_1 and u_2, the equations which correspond to (18) are

$$X_s \delta t_1 = \frac{\partial u_1}{\partial x_s} \delta \alpha_1 + \frac{\partial u_2}{\partial x_s} \delta \alpha_2 + t_1 \sum_{m=1}^{2n} a_{m,s} \delta x_m.$$

There are three independent variables t_1, α_1, α_2, which will require three of these $2n$ equations for their determination by quadratures. Of the remaining $2n-3$ equations, only $2n-5$ are independent determining, as they do, the quantities $u_3, \ldots, u_n, \frac{\alpha_4}{\alpha_3}, \ldots, \frac{\alpha_n}{\alpha_3}$; and since there are three independent variables, each of the equations leads to three equations in the forms

$$X_s = t_1 \sum_{m=1}^{2n} a_{m,s} \frac{\partial x_m}{\partial t_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

$$0 = \frac{\partial u_1}{\partial x_s} + t_1 \sum_{m=1}^{2n} a_{m,s} \frac{\partial x_m}{\partial \alpha_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)},$$

$$0 = \frac{\partial u_2}{\partial x_s} + t_1 \sum_{m=1}^{2n} a_{m,s} \frac{\partial x_m}{\partial \alpha_2} \quad \ldots\ldots\ldots\ldots\ldots\text{(iii)}.$$

Hence there are three systems of subsidiary equations, each system containing $2n-5$ independent members; and since u_1 and u_2 are constant and t_1, α_1, α_2 are determined in terms of the variables x, it follows that the equations subsist in $2n-4$ $(=2n+1-5)$ variables other than those three independent variables which do not enter into the expression of their integrals.

The condition at the end of § 103 that u_2 should be independent of α_1, viz.,

$$0 = \sum_{m=1}^{2n} \sum_{s=1}^{2n} \frac{\partial u_1}{\partial x_s} R_{m,s} \frac{\partial u_2}{\partial x_m},$$

is easily proved to be the condition that u_1 should be independent of α_2. And it is easy to verify, as there, that a third integral of the original differential equation satisfies the relations

$$\sum_{m=1}^{2n} \sum_{s=1}^{2n} X_s R_{m,s} \frac{\partial u_3}{\partial x_m} = 0,$$

$$\sum_{m=1}^{2n} \sum_{s=1}^{2n} \frac{\partial u_1}{\partial x_s} R_{m,s} \frac{\partial u_3}{\partial x_m} = 0,$$

$$\sum_{m=1}^{2n} \sum_{s=1}^{2n} \frac{\partial u_2}{\partial x_s} R_{m,s} \frac{\partial u_3}{\partial x_m} = 0,$$

which agree with those given (§ 122) by Clebsch's theory.

And so on, either for any number of independent integrals of the original subsidiary system supposed known; or for any number of the integrals of the differential equation supposed known.

CHAPTER VII.

APPLICATION TO PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER.

106. As Pfaff's investigations were originally initiated with a view to the solution of partial differential equations of the first order, it is of some interest to indicate briefly the form which the solution takes when deduced by the theory of Pfaffian equations.

Let

$$f(z, x_1, \ldots, x_n, p_1, \ldots, p_n) = a \quad \ldots\ldots\ldots\ldots\ldots\text{(i)}$$

be any differential equation, the integral of which is required. We have always

$$-dz + p_1 dx_1 + \ldots\ldots + p_n dx_n = 0 \quad \ldots\ldots\ldots\ldots\text{(ii)},$$

which, in connection with the equation (i), may be regarded in two views.

First, we may by (i) find the value of any one of the quantities $z, x_1, \ldots, x_n, p_1, \ldots, p_n$ in terms of the remainder, say

$$p_n = \theta(z, x_1, \ldots, x_n, p_1, \ldots, p_{n-1}, a) = \theta;$$

then (ii) becomes

$$-dz + p_1 dx_1 + \ldots\ldots + p_{n-1} dx_{n-1} + \theta dx_n = 0 \ldots\ldots\text{(iii)},$$

a Pfaffian equation in $2n$ variables $z, x_1, \ldots, x_n, p_1, \ldots, p_{n-1}$. This is the view in which Pfaff regarded the question.

Secondly, we may consider (ii) as a Pfaffian equation in $2n+1$ variables $z, x_1, \ldots, x_n, p_1, \ldots, p_n$; it is known that one of the integrals of the equivalent system may be assumed at will (§ 69), and so equation (i) is taken to be that assumed integral. This is the view in which Natani regarded the question.

These two views will be taken in turn, so as to exhibit the subsidiary system of equations, which are materially simpler than in the general case owing to the zero values of the coefficients of the elements dp, and so as to indicate the solution of the equation.

Clebsch's method is avowedly the generalisation of Jacobi's method for partial differential equations of the first order to the Pfaffian problem; and so his method does not in itself furnish any advance beyond the general Jacobian theory, that is, no advance along the lines of Pfaff's equation.

The application of Lie's method, practically repeated by Darboux, is given as an example (§ 136); and the method of Frobenius is entirely limited to the theory of the transformation of Pfaffian expressions.

107. To obtain the Pfaffian solution of the given differential equation we take (iii), which, when written in the form

$$p_1 dx_1 + p_2 dx_2 + \ldots + p_{n-1} dx_{n-1} + \theta dx_n + 0 . dp_1 + \ldots + 0 . dp_{n-1} + (-1) dz = 0,$$

will agree with the general Pfaffian equation considered in the preceding chapters, if in the latter we make

$$\begin{aligned} x_{2n} &= z; & x_{n+r} &= p_r, \text{ for } r = 1, \ldots, n-1; \\ X_{2n} &= -1; & X_{n+r} &= 0, \text{ for } r = 1, \ldots, n-1; \\ X_n &= \theta; & X_r &= p_r, \text{ for } r = 1, \ldots, n-1. \end{aligned}$$

Then to form the subsidiary equations (§ 55) we need the quantities $a_{i,j}$. These are easily found to be

$$a_{i,j} = 0, \text{ if neither } i \text{ nor } j \text{ be greater than } n-1;$$

$$a_{n,i} = \frac{\partial \theta}{\partial x_i}, \text{ for } i = 1, \ldots, n-1;$$

$$a_{q,q} = 0 \text{ always, for all values of } q;$$

$$a_{n,n+r} = \frac{\partial \theta}{\partial p_r}, \text{ for } r = 1, \ldots, n-1;$$

$$a_{n,2n} = \frac{\partial \theta}{\partial z};$$

$$a_{n+r,r} = -1, \text{ for } r = 1, \ldots, n-1;$$

$$a_{n+r,i} = 0, \text{ if } i \text{ and } r \text{ be different, for } i = 1, \ldots, n-1;$$

$$a_{n+r,n+s} = 0, \text{ for } s = 1, \ldots, n \text{ and } r = 1, \ldots, n;$$

$$a_{2n,i} = 0, \text{ for } i = 1, \ldots, n-1.$$

Then the equations already referred to take the form

$$X_i = -y_n \frac{\partial\theta}{\partial x_i} + y_{n+i}, \text{ for } i = 1, \ldots, n-1;$$

$$X_{n+r} = -y_r - y_n \frac{\partial\theta}{\partial p_r}, \text{ for } r = 1, \ldots, n-1;$$

$$X_n = \sum_{i=1}^{n-1} y_i \frac{\partial\theta}{\partial x_i} + \sum_{r=1}^{n-1} y_{n+r} \frac{\partial\theta}{\partial p_r} + y_{2n} \frac{\partial\theta}{\partial z};$$

$$X_{2n} = -y_n \frac{\partial\theta}{\partial z}.$$

Now since $\theta = p_n$ is derived from the equation (i), we have

$$\frac{\partial f}{\partial p_n}\frac{\partial\theta}{\partial x_i} + \frac{\partial f}{\partial x_i} = 0, \text{ for } i = 1, \ldots, n;$$

$$\frac{\partial f}{\partial p_n}\frac{\partial\theta}{\partial p_r} + \frac{\partial f}{\partial p_r} = 0, \text{ for } r = 1, \ldots, n;$$

$$\frac{\partial f}{\partial p_n}\frac{\partial\theta}{\partial z} + \frac{\partial f}{\partial z} = 0.$$

Substituting from these equations for the derivatives of θ, inserting the values of the quantities X, and bearing in mind the definitions (equations (9) of § 55) of the quantities y, we have

$$\frac{dx_1}{\dfrac{\partial f}{\partial p_1}} = \ldots\ldots = \frac{dx_n}{\dfrac{\partial f}{\partial p_n}} = \frac{dz}{P} = \frac{dp_1}{P_1} = \ldots\ldots = \frac{dp_n}{P_n},^*$$

where

$$P_i = -\frac{\partial f}{\partial x_i} - p_i \frac{\partial f}{\partial z}, \text{ for } i = 1, \ldots, n;$$

$$P = \sum_{i=1}^{n} p_i \frac{\partial f}{\partial p_i},$$

and the value of μ is

$$-\frac{1}{P}\frac{\partial f}{\partial z}.$$

108. In the Pfaffian process of solution, it is necessary to integrate the subsidiary system just found and use the $2n-1$

* These are the subsidiary equations for the derivation of the first integral with the form of equation given in the example in § 213, *Treatise*; they of course correspond to the subsidiary system of the simpler form of equation considered in § 215, *Treatise*.

integrals other than $f=a$ to transform the original equation. One of the integrals thus used is taken as an integral of the original equation; and so the modified equation comes to be an equation in $2n-2$ variables.

We proceed now as in the general case already considered. It is thus necessary to integrate n subsidiary systems of equations; and each subsidiary system leads to one integral of the differential equation. We shall thus ultimately obtain n integrals, which will involve the variables $z, x_1, \ldots, x_n, p_1, \ldots, p_n$ and n arbitrary constants; when from these n equations, taken with $f=a$, the n quantities $p_1, \ldots, p_n$ are eliminated, the result is an equation between $z, x_1, \ldots, x_n$ and n arbitrary constants, which is the Complete Integral of the equation.

109. The last section requires that n sets of subsidiary equations shall be integrated. A great simplification was made by Jacobi, so that the integration of the first system alone is necessary: this was effected by the introduction of "initial values" of the variables*—a step connected with the construction of the principal integrals of a set of differential equations.

Let

$$u_1 = a_1,\ u_2 = a_2,\ \ldots\ldots,\ u_{2n-1} = a_{2n-1}$$

be a set of $2n-1$ independent integrals of the subsidiary system, which may be taken in the form

$$P\frac{\partial x_r}{\partial z} = \frac{\partial f}{\partial p_r}, \text{ for } r = 1, \ldots, n,$$

$$P\frac{\partial p_r}{\partial z} = -\frac{\partial f}{\partial x_r} - p_r\frac{\partial f}{\partial z}, \text{ for } r = 1, \ldots, n;$$

* See the memoir "Ueber die Reduction der Integration der partiellen Differentialgleichungen erster Ordnung zwischen irgend einer Zahl Variabeln auf die Integration eines einzigen Systemes gewöhnlicher Differentialgleichungen," *Crelle*, t. xvii. (1837) pp. 97—162, especially § 9, pp. 136 sqq.; or in the Collected Works, vol. iv., pp. 57—127, especially pp. 100 sqq.

The idea of introducing these initial values is there assigned to the then recent investigations of Hamilton on dynamics; and, in consequence, the method is often called the Jacobi-Hamiltonian method. It is however pointed out by Lie, *Math. Ann.* t. viii., p. 215 (note) and by Mansion, "Théorie des équations aux dérivées partielles du premier ordre," p. 115 (note) that the method is really due to Cauchy, who had published it in 1819: see also Cauchy's "Exercices d'Analyse et de Physique Mathématique," t. ii., pp. 270—272.

the remaining integral of the system being the given equation

$$f = a.$$

Take z, u_1, ..., u_{2n-1} as a new system of variables and express x_1, ..., x_n, p_1, ..., p_n in terms of a, z, u_1, ..., u_{2n-1}; then, when the values are substituted in the foregoing subsidiary system, those equations are identities. Now since

$$P = p_1 \frac{\partial f}{\partial p_1} + \dots\dots + p_n \frac{\partial f}{\partial p_n},$$

we have from the first n equations

$$1 = p_1 \frac{\partial x_1}{\partial z} + \dots\dots + p_n \frac{\partial x_n}{\partial z},$$

which with the substituted values is an identity. It gives on differentiation

$$0 = \frac{\partial p_1}{\partial u} \frac{\partial x_1}{\partial z} + \dots\dots + \frac{\partial p_n}{\partial u} \frac{\partial x_n}{\partial z}$$

$$+ p_1 \frac{\partial^2 x_1}{\partial u \, \partial z} + p_2 \frac{\partial^2 x_2}{\partial u \, \partial z} + \dots\dots + p_n \frac{\partial^2 x_n}{\partial u \, \partial z}$$

for each of the quantities u; and this by means of the first n equations leads to

$$\frac{\partial f}{\partial p_1} \frac{\partial p_1}{\partial u} + \dots\dots + \frac{\partial f}{\partial p_n} \frac{\partial p_n}{\partial u} = -P \left(p_1 \frac{\partial^2 x_1}{\partial u \, \partial z} + \dots\dots + p_n \frac{\partial^2 x_n}{\partial u \, \partial z} \right).$$

Again the equation $f = a$ is an identity after the values are substituted; and therefore

$$0 = \frac{\partial f}{\partial p_1} \frac{\partial p_1}{\partial u} + \dots\dots + \frac{\partial f}{\partial p_n} \frac{\partial p_n}{\partial u} + \frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial u} + \dots\dots + \frac{\partial f}{\partial x_n} \frac{\partial x_n}{\partial u}$$

(bearing in mind that u_1, ..., u_{2n-1}, z are the $2n$ independent variables) or, by the use of the last n equations of the subsidiary system considered as giving the quantities $\frac{\partial f}{\partial x}$, we have

$$\frac{\partial f}{\partial p_1} \frac{\partial p_1}{\partial u} + \dots\dots + \frac{\partial f}{\partial p_n} \frac{\partial p_n}{\partial u}$$

$$= \frac{\partial x_1}{\partial u} \left(P \frac{\partial p_1}{\partial z} + p_1 \frac{\partial f}{\partial z} \right) + \dots\dots + \frac{\partial x_n}{\partial u} \left(P \frac{\partial p_n}{\partial z} + p_n \frac{\partial f}{\partial z} \right).$$

Equating the two expressions obtained for $\Sigma \dfrac{\partial f}{\partial p}\dfrac{\partial p}{\partial u}$, we have

$$P\left\{\left(p_1\frac{\partial^2 x_1}{\partial u\,\partial z}+\frac{\partial x_1}{\partial u}\frac{\partial p_1}{\partial z}\right)+\ldots\ldots+\left(p_n\frac{\partial^2 x_n}{\partial u\,\partial z}+\frac{\partial x_n}{\partial u}\frac{\partial p_n}{\partial z}\right)\right\}$$
$$+\frac{\partial f}{\partial z}\left(p_1\frac{\partial x_1}{\partial u}+\ldots\ldots+p_n\frac{\partial x_n}{\partial u}\right)=0;$$

and therefore on integration

$$p_1\frac{\partial x_1}{\partial u}+\ldots\ldots+p_n\frac{\partial x_n}{\partial u}=Ce^{-\int^z \frac{\partial f}{\partial z}\frac{dz}{P}},$$

where C is a quantity independent of z.

It is at this stage that the "initial values" are introduced. Since C is independent of z, its value will be unaltered, if any special value be assigned to z, say a zero value. Let the values of the variables x for this value of z be $\xi_1, \ldots, \xi_n$ and those of the variables p be $\pi_1, \ldots, \pi_n$, all of which will be functions of $u_1, \ldots, u_{2n-1}$ determined by the $2n$ independent equations

$$\left.\begin{array}{l} f(0, \xi_1, \ldots, \xi_n, \pi_1, \ldots, \pi_n)=a \\ u_i(0, \xi_1, \ldots, \xi_n, \pi_1, \ldots, \pi_n)=a_i=u_i \end{array}\right\},$$

for $i=1, \ldots, 2n-1$. The forms of the result are

$$\begin{aligned}\xi_i &= \text{function } (a, u_1, \ldots, u_{2n-1})\\ &= \text{function } (z, x_1, \ldots, x_n, p_1, \ldots, p_n),\end{aligned}$$

when for a we substitute $f(z, x_1, \ldots, x_n, p_1, \ldots, p_n)$ and for u_i its value in terms of z, x, p; and the form of the function is such that, when z is made zero, it reduces to x_i.

To determine C we insert these simultaneous values in the equation above obtained, assigning zero (because it is the special value of z) as the lower limit of the integral in the exponential term; and we find

$$\pi_1\frac{\partial \xi_1}{\partial u}+\ldots\ldots+\pi_n\frac{\partial \xi_n}{\partial u}=C,$$

so that

$$p_1\frac{\partial x_1}{\partial u}+\ldots\ldots+p_n\frac{\partial x_n}{\partial u}=\left(\pi_1\frac{\partial \xi_1}{\partial u}+\ldots\ldots+\pi_n\frac{\partial \xi_n}{\partial u}\right)e^{-\int_0^z \frac{\partial f}{\partial z}\frac{dz}{P}}$$

Substituting now for $x_1, \ldots, x_n$ their values in terms of $z, u_1, \ldots, u_{2n-1}$, we have

$$- dz + p_1 dx_1 + \ldots\ldots + p_n dx_n$$
$$= dz\left(-1 + p_1 \frac{\partial x_1}{\partial z} + \ldots\ldots + p_n \frac{\partial x_n}{\partial z}\right) + \sum_{i=1}^{n} \sum_{j=1}^{2n-1} p_i \frac{\partial x_i}{\partial u_j} du_j.$$

The left-hand side is zero: the coefficient of dz on the right-hand side vanishes, as indeed it ought because we are carrying out a Pfaffian reduction; and so the original equation is replaced by

$$\sum_{i=1}^{n} \sum_{j=1}^{2n-1} p_i \frac{\partial x_i}{\partial u_j} du_j = 0.$$

Substituting for $\sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial u}$ from above, and rejecting the exponential factor which does not furnish a solution of the equation, we have

$$\sum_{j=1}^{2n-1} \left(\pi_1 \frac{\partial \xi_1}{\partial u_j} + \ldots\ldots + \pi_n \frac{\partial \xi_n}{\partial u_j}\right) du_j = 0.$$

The quantities ξ_i in this equation are functions only of $a, u_1, \ldots, u_{2n-1}$, and so also are the quantities π; hence the new equation involves only $2n-1$ variables and is thus the transformation of the original Pfaffian equation. But, further, since ξ is a function only of the variables u, we have

$$\frac{\partial \xi}{\partial u_1} du_1 + \ldots\ldots + \frac{\partial \xi}{\partial u_{2n-1}} du_{2n-1} = d\xi,$$

and therefore the equation is

$$\pi_1 d\xi_1 + \pi_2 d\xi_2 + \ldots\ldots + \pi_n d\xi_n = 0,$$

that is, *it is the normal reduced form equivalent to the original Pfaffian equation.*

We have already seen (§ 69) that an integral system of this is

$$\xi_1 = c_1,\ \xi_2 = c_2,\ \ldots\ldots,\ \xi_n = c_n,$$

where the quantities c are constants; and therefore we infer the following result:—

To integrate the equation

$$f(z, x_1, \ldots, x_n, p_1, \ldots, p_n) = a$$

form the subsidiary system of $2n$ *ordinary equations*

$$\frac{dx_1}{\dfrac{\partial f}{\partial p_1}} = \ldots\ldots = \frac{dx_n}{\dfrac{\partial f}{\partial p_n}} = \frac{dz}{P} = \frac{dp_1}{P_1} = \ldots\ldots = \frac{dp_n}{P_n},$$

where

$$P_i = -\frac{\partial f}{\partial x_i} - p_i \frac{\partial f}{\partial z}, \textit{for } i = 1, \ldots, n \textit{; and}$$

$$P = \sum_{i=1}^{n} p_i \frac{\partial f}{\partial p_i}.$$

Let a set of $2n-1$ *integrals, independent of one another and of* $f = a$ *(which with* $f = a$ *make up a complete system of integrals of the subsidiary system) be*

$$u_s = u_s(z, x_1, \ldots, x_n, p_1, \ldots, p_n) = \textit{constant} = a_s,$$

for $s = 1, 2, \ldots, 2n-1$; *and solve the* $2n$ *equations*

$$f(0, \xi_1, \ldots, \xi_n, \pi_1, \ldots, \pi_n) = a,$$
$$u_s(0, \xi_1, \ldots, \xi_n, \pi_1, \ldots, \pi_n) = a_s,$$

so as to give specially the n *quantities* ξ *in the forms*

$$\xi_r = \phi_r(a, a_1, \ldots, a_{2n-1}),$$

for $r = 1, \ldots, n$. *Then the elimination of* $p_1, \ldots, p_n$ *among the* $n+1$ *equations*

$$f(z, x_1, \ldots, x_n, p_1, \ldots, p_n) = a,$$
$$\phi_r(a, u_1, \ldots\ldots\ldots\ldots\ldots, u_{2n-1}) = a_r,$$

after we have substituted in ϕ_r *for the quantities* u *their values* $u_r(z, x_1, \ldots, x_n, p_1, \ldots, p_n)$, *will lead to an equation involving* $z, x_1, \ldots, x_n$ *and the* n *arbitrary constants* a_r. *This is the Complete Integral of the original differential equation.*

110. The form, to which the preceding consideration of the relation between $f = a$ and $dz = \sum_{i=1}^{n} p_i dx_i$ would lead when Natani's method is adopted for the reduction, has been most briefly referred to in § 92. The result is equivalent to that which precedes and may be enunciated as follows:—

Let the $2n-1$ *independent integrals of the subsidiary system, other than* $f = a$, *be*

$$u_i(z, x_1, \ldots, x_n, p_1, \ldots, p_n) = \textit{constant},$$

for values $1, 2, \ldots, 2n-1$ *of* i. *Then the elimination of* $p_1, \ldots, p_n, \pi_1, \ldots, \pi_n$ *from the* $2n+1$ *equations*

$$f(z, x_1, \ldots, x_n, p_1, \ldots, p_n) = a,$$
$$f(0, \alpha_1, \ldots, \alpha_n, \pi_1, \ldots, \pi_n) = a,$$
$$u_i(z, x_1, \ldots, x_n, p_1, \ldots, p_n) = u_i(0, \alpha_1, \ldots, \alpha_n, \pi_1, \ldots, \pi_n),$$

(where $i = 1, \ldots, 2n-1$*), leads to an equation between* $z, x_1, \ldots, x_n$ *which involves* n *arbitrary constants* $\alpha_1, \ldots, \alpha_n$ *and is the Complete Integral of the differential equation.*

111. We now proceed to the consideration of the alternative relation between $f = a$ and the equation

$$-dz + p_1 dx_1 + \ldots\ldots + p_n dx_n = 0,$$

in which the equation among the differential elements is regarded as an equation in $2n+1$ variables, and has $f = a$ for that integral which can be assumed arbitrarily.

The foregoing equation agrees with

$$\sum_{i=1}^{2n+1} X_i dx_i = 0,$$

if we take

$$x_{n+r} = p_r, \text{ for } r = 1, \ldots, n; \quad x_{2n+1} = z;$$
$$X_{n+r} = 0, \quad \text{for } r = 1, \ldots, n; \; X_{2n+1} = -1;$$
$$X_s = p_s, \quad \text{for } s = 1, \ldots, n.$$

And then we have

$a_{ms} = 0$, for m and $s = 1, \ldots, n$;

$a_{m, n+m} = 1$, for $m = 1, \ldots, n$;

$a_{m, n+s} = 0$, for $m = 1, \ldots, n$ and $s = 1, \ldots, n+1$, if m and s be unequal;

$a_{m+r, n+s} = 0$, for r and $s = 1, \ldots, n+1$.

Then constructing the subsidiary equations as in §§ 93 and 102 and taking by preference the later form, we have as the complete system

$$p_s \delta t_1 = \frac{\partial f}{\partial x_s} \delta\mu - t_1 \delta p_s, \text{ for } s = 1, \ldots, n;$$

$$0 = \frac{\partial f}{\partial p_s} \delta\mu + t_1 \delta x_s, \text{ for } s = 1, \ldots, n;$$

$$-\delta t_1 = \frac{\partial f}{\partial z} \delta\mu,$$

there being only one independent variable.

If $\dfrac{\partial f}{\partial z} = 0$ so that the given differential equation is explicitly

independent of z, then $\delta t_1 = 0$ and the subsidiary equations take the simple (Hamiltonian) form

$$\ldots = \frac{dx_s}{-\frac{\partial f}{\partial p_s}} = \ldots = \frac{dp_r}{\frac{\partial f}{\partial x_r}} = \ldots;$$

if $\frac{\partial f}{\partial z}$ be not zero, then the equations take the form

$$\ldots = \frac{dx_s}{-\frac{\partial f}{\partial p_s}} = \ldots = \frac{dp_r}{\frac{\partial f}{\partial x_r} + p_r \frac{\partial f}{\partial z}} = \ldots;$$

the equations in each case being associated with $f = a$.

There are thus $2n - 1$ equations in all in the subsidiary system and so there must be $2n - 1$ integrals; and $f = a$ is not an integral of the system unless we associate $dz = \sum_{i=1}^{n} p_i dx_i$ with it. There are then $2n$ equations in the system, and it has $2n$ integrals. Taking the principal integrals, and denoting them by $x_1', \ldots, x_n', p_1', \ldots, p_n'$ for $z = 0$, we have by § 94

$$A\left(-dz + \sum_{i=1}^{n} p_i dx_i\right) = \mu df + A' \sum_{i=1}^{n} p_i' dx_i',$$

or, since $f = a$ permanently,

$$-dz + \sum_{i=1}^{n} p_i dx_i = \frac{A'}{A} \sum_{i=1}^{n} p_i' dx_i';$$

and therefore the differential equation is replaced by

$$\sum_{i=1}^{n} p_i' dx_i' = 0,$$

an integral system of which is

$$x_1' = c_1, \quad \ldots, \quad x_n' = c_n.$$

This leads to the same result as in § 110.

If $n = 2$, the subsidiary system is

$$\frac{dx_1}{-\frac{\partial f}{\partial p_1}} = \frac{dx_2}{-\frac{\partial f}{\partial p_2}} = \frac{dp_1}{\frac{\partial f}{\partial x_1} + p_1 \frac{\partial f}{\partial z}} = \frac{dp_2}{\frac{\partial f}{\partial x_2} + p_2 \frac{\partial f}{\partial z}} = \frac{dz}{-p_1 \frac{\partial f}{\partial p_1} - p_2 \frac{\partial f}{\partial p_2}};$$

each of these fractions being equal to

$$-\frac{d\mu}{t_1} \text{ and to } \frac{dt_1}{t_1 \frac{\partial f}{\partial z}},$$

for this special case as for the general value of n.

When the integrals, three in addition to $f=a$, of the main part of the subsidiary system are known, then t_1 is determined by a quadrature and thence A (and so A', by substitution in A of the principal integrals); and, t_1 being known, μ is determinable by a quadrature.

In actual practice, however, these quadratures may be dispensed with, if only the actual result be desired without the explicit form of all the intermediate stages. For example, taking the equation

$$p_1^2+p_2^2+2z+x_1^2+x_2^2=a,$$

the subsidiary system is

$$\frac{dx_1}{-p_1}=\frac{dx_2}{-p_2}=\frac{dp_1}{x_1+p_1}=\frac{dp_2}{x_2+p_2}=\frac{dz}{-p_1^2-p_2^2}.$$

Three integrals of the system other than the given one are

$$p_2-\omega x_2=A\,(p_1-\omega x_1),$$
$$(\omega p_1-x_1)^{\omega}=B\,(p_1-\omega x_1),$$
$$(\omega p_2-x_2)^{\omega}=C\,(p_1-\omega x_1),$$

where ω is a cube root of unity and A, B, C are constants. Then it is easy to see that the Complete Integral of the given differential equation is obtained by the elimination of p_1, p_2, π_1, π_2 among the equations

$$p_1^2+p_2^2+2z+x_1^2+x_2^2=a,$$
$$\pi_1^2+\pi_2^2+\alpha^2+\beta^2=a,$$
$$\frac{p_2-\omega x_2}{p_1-\omega x_1}=\frac{\pi_2-\omega\beta}{\pi_1-\omega\alpha},$$
$$\frac{(\omega p_1-x_1)^{\omega}}{p_1-\omega x_1}=\frac{(\omega\pi_1-\alpha)^{\omega}}{\pi_1-\omega\alpha},$$
$$\frac{(\omega p_2-x_2)^{\omega}}{p_1-\omega x_1}=\frac{(\omega\pi_2-\beta)^{\omega}}{\pi_1-\omega\alpha},$$

where α and β are arbitrary constants.

Ex. Treat similarly the equations*:—

(i) $(1+p_1^2+p_2^2)\,z^2=a$;

(ii) $ax_1+bz-p_1+f(x_1,\ p_2)=0$.

112. Natani also indicated in connection with § 33 what is the first step towards the extension of the preceding method to the integration of a system of simultaneous partial differential equations of the first order. A sufficient illustration will be given by supposing a system of two such equations

$$f=a,\quad u_1=b.$$

* These are taken from Natani's "Die höhere Analysis" (1866), pages 341—353 of which deal specially with the integration of partial differential equations of the first order.

Then proceeding as in the corresponding case for the total equation, we find that the equations, which are subsidiary to the equation

$$A\{-dz + \sum_{i=1}^{n} p_i dx_i\} = \mu df + \sum_{i=1}^{n} a_i du_i$$
$$= \mu df + a_1 du_1 + \sum_{i=2}^{n} a_i du_i,$$

take the form

$$\left.\begin{aligned} p_s \delta t_1 &= \frac{\partial f}{\partial x_s}\delta\mu + \frac{\partial u_1}{\partial x_s}\delta a_1 - t_1 \delta p_s \\ 0 &= \frac{\partial f}{\partial p_s}\delta\mu + \frac{\partial u_1}{\partial p_s}\delta a_1 + t_1 \delta x_s \\ -\delta t_1 &= \frac{\partial f}{\partial z}\delta\mu + \frac{\partial u_1}{\partial z}\delta a_1 \end{aligned}\right\},$$

the first two of these holding for $s = 1, \ldots, n$. In this system there are two independent variables, which may be taken as μ and a_1.

Writing

$$\frac{\partial}{\partial x_s} + p_s \frac{\partial}{\partial z} = \frac{d}{dx_s},$$

the elimination of δt_1 leads to

$$\left.\begin{aligned} 0 &= \frac{df}{dx_s}\delta\mu + \frac{du_1}{dx_s}\delta a_1 - t_1 \delta p_s \\ 0 &= \frac{\partial f}{\partial p_s}\delta\mu + \frac{\partial u_1}{\partial p_s}\delta a_1 + t_1 \delta x_s \end{aligned}\right\}.$$

The existence of these equations implies a certain relation between f and u_1; for we have

$$\left(\frac{du_1}{dx_s}\frac{\partial f}{\partial p_s} - \frac{df}{dx_s}\frac{\partial u_1}{\partial p_s}\right)\delta a_1 = \frac{df}{dx_s}\delta x_s + \frac{\partial f}{\partial p_s}\delta p_s,$$

and therefore

$$\sum_{s=1}^{n}\left(\frac{du_1}{dx_s}\frac{\partial f}{\partial p_s} - \frac{df}{dx_s}\frac{\partial u_1}{\partial p_s}\right)\delta a_1 = \sum_{s=1}^{n}\left(\frac{df}{dx_s}\delta x_s + \frac{\partial f}{\partial p_s}\delta p_s\right)$$
$$= df = 0,$$

so that

$$\sum_{s=1}^{n}\left(\frac{du_1}{dx_s}\frac{\partial f}{\partial p_s} - \frac{df}{dx_s}\frac{\partial u_1}{\partial p_s}\right) = 0$$

is a necessary relation*.

* This is not explicitly given by Natani, either in his memoir or in his treatise.

Let us assume that this condition is satisfied. Then each of the foregoing equations leads to two: and so the new subsidiary equations are composed of the independent equations of the sets

$$\left.\begin{aligned}\frac{df}{dx_s} - t_1\frac{\partial p_s}{\partial \mu} = 0\\ \frac{\partial f}{\partial p_s} + t_1\frac{\partial x_s}{\partial \mu} = 0\end{aligned}\right\},\qquad \left.\begin{aligned}\frac{du_1}{dx_s} - t_1\frac{\partial p_s}{\partial \alpha_1} = 0\\ \frac{\partial u_1}{\partial p_s} + t_1\frac{\partial x_s}{\partial \alpha_1} = 0\end{aligned}\right\},$$

which are two systems of equations. Then, as in § 103, a complete system of integrals is given by $\alpha_1, u_2, \ldots, u_n, \frac{\alpha_3}{\alpha_2}, \ldots, \frac{\alpha_n}{\alpha_2}$; and any simultaneous integral of the two systems is a function of these integrals.

Let such a simultaneous integral be θ; since it satisfies the first system we have, as above,

$$\sum_{s=1}^{n}\left(\frac{d\theta}{dx_s}\frac{\partial f}{\partial p_s} - \frac{df}{dx_s}\frac{\partial \theta}{\partial p_s}\right) = 0.$$

Now suppose θ expressed as a function of the system $\alpha_1, u_2, \ldots, u_n, \frac{\alpha_3}{\alpha_2}, \ldots, \frac{\alpha_n}{\alpha_2}$; then we have

$$\begin{aligned}\frac{\partial\theta}{\partial\alpha_1} &= \sum_{s=1}^{n}\frac{\partial\theta}{\partial x_s}\frac{\partial x_s}{\partial\alpha_1} + \sum_{s=1}^{n}\frac{\partial\theta}{\partial p_s}\frac{\partial p_s}{\partial\alpha_1} + \frac{\partial\theta}{\partial z}\frac{\partial z}{\partial\alpha_1}\\ &= \sum_{s=1}^{n}\left(\frac{\partial\theta}{\partial x_s} + p_s\frac{\partial\theta}{\partial z}\right)\frac{\partial x_s}{\partial\alpha_1} + \sum_{s=1}^{n}\frac{\partial\theta}{\partial p_s}\frac{\partial p_s}{\partial\alpha_1},\end{aligned}$$

for we have in the subsidiary system the implicit equation

$$\frac{\partial z}{\partial\alpha_1} = \sum_{s=1}^{n} p_s\frac{\partial x_s}{\partial\alpha_1},$$

and therefore

$$\frac{\partial\theta}{\partial\alpha_1} = \frac{1}{t_1}\sum_{s=1}^{n}\left(\frac{\partial\theta}{\partial p_s}\frac{du_1}{dx_s} - \frac{\partial u}{\partial p_s}\frac{d\theta}{dx_s}\right).$$

If θ be an integral of the required differential equation, then it must be independent of α_1 and the right-hand side will vanish.

There is however no occasion that it should be, for he regards $u_1 = b$ as an *integral* of the former subsidiary system and is investigating the effect of this knowledge in diminishing the amount of integration required for that subsidiary system as in § 103.

The equivalent of the foregoing relation in regard to a *second integral* u_2 is explicitly given by him.

If the right-hand side do not vanish, then we have the same series of alternatives for the subsidiary system as occurred in § 104; and, *mutatis mutandis,* the discussion there given is applicable here.

Ex. Integrate, by Pfaff's process, the simultaneous equations

$$p_1 = p_2 z = p_3 z^2,$$

where the variable z is dependent upon three variables x_1, x_2, x_3, the quantities p_1, p_2, p_3 being its derivatives with regard to those variables.

Also, with similar notation, the equations

$$p_1 = p_2 z = p_3 z^2 = p_4 z^3.$$

(Raabe.)

CHAPTER VIII.

CLEBSCH'S METHOD.

113. It has already been seen, in § 68, that an expression which contains $2n$ or $2n-1$ differential elements can be reduced to one which contains not more than n such elements; but that, at each stage of the method of reduction there used, alternatives are possible and therefore any reduced form so obtained is not unique.

When, however, one reduced form has been obtained, all others can be deduced from it by the following process, which constitutes the generalisation of any special solution of Pfaff's problem.

Let the smallest possible number of differential elements in the reduced form of a given expression be m; and let such a reduced form be

$$F_1 df_1 + F_2 df_2 + \ldots\ldots + F_m df_m,$$

where there is no identical relation among the quantities F and f. Let another (and therefore* an equivalent) reduced form be

$$\Phi_1 d\phi_1 + \Phi_2 d\phi_2 + \ldots\ldots + \Phi_m d\phi_m,$$

with a similar absence of any identical relation among the quantities Φ and ϕ. Then we have

$$\sum_{\lambda=1}^{m} F_\lambda df_\lambda = \sum_{\mu=1}^{m} \Phi_\mu d\phi_\mu;$$

and therefore, as quantities on one side of the equation are independent of one another, we may consider the $2m$ quantities Φ and ϕ as functions of the $2m$ independent quantities F and f, and as determined by the equations

$$\left.\begin{aligned} F_i &= \Phi_1 \frac{\partial \phi_1}{\partial f_i} + \Phi_2 \frac{\partial \phi_2}{\partial f_i} + \ldots\ldots + \Phi_m \frac{\partial \phi_m}{\partial f_i} \\ 0 &= \Phi_1 \frac{\partial \phi_1}{\partial F_j} + \Phi_2 \frac{\partial \phi_2}{\partial F_j} + \ldots\ldots + \Phi_m \frac{\partial \phi_m}{\partial F_j} \end{aligned}\right\} \ldots\ldots\ldots(1)$$

* See § 142, post.

for $i=1, 2, \ldots, m$ and $j=1, 2, \ldots, m$. When the coefficients Φ are eliminated from the second set of m equations, the result is

$$\frac{\partial(\phi_1, \phi_2, \ldots, \phi_m)}{\partial(F_1, F_2, \ldots, F_m)}=0.$$

Now the m quantities ϕ are functions of $F_1, F_2, \ldots, F_m, f_1, \ldots, f_m$, and the last equation shews that, from among the m equations which express this functionality, the m quantities F can be eliminated; the result is thus of the form

$$\Pi(\phi_1, \phi_2, \ldots, \phi_m, f_1, f_2, \ldots, f_m)=0 \quad\ldots\ldots\ldots\ldots(2).$$

Moreover, since the only limitation is that imposed by the above Jacobian, this function Π can be taken quite arbitrary.

In equation (2) the quantities F occur only implicitly through their introduction by the quantities ϕ: hence for each of the indices $j=1, 2, \ldots, m$ we have

$$\frac{\partial\Pi}{\partial\phi_1}\frac{\partial\phi_1}{\partial F_j}+\frac{\partial\Pi}{\partial\phi_2}\frac{\partial\phi_2}{\partial F_j}+\ldots\ldots+\frac{\partial\Pi}{\partial\phi_m}\frac{\partial\phi_m}{\partial F_j}=0,$$

whence from a comparison with the second set of m equations in (1) we have

$$\Phi_r=\lambda\frac{\partial\Pi}{\partial\phi_r} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

(for $r=1, 2, \ldots, m$), where λ is an undetermined factor. Again, in equation (2) the quantities f occur both implicitly, through their introduction by the quantities ϕ, and also explicitly; so that

$$\frac{\partial\Pi}{\partial\phi_1}\frac{\partial\phi_1}{\partial f_i}+\frac{\partial\Pi}{\partial\phi_2}\frac{\partial\phi_2}{\partial f_i}+\ldots\ldots+\frac{\partial\Pi}{\partial\phi_m}\frac{\partial\phi_m}{\partial f_i}+\frac{\partial\Pi}{\partial f_i}=0.$$

Hence, from the first set of m equations in (1), we have

$$\begin{aligned}F_i&=\Phi_1\frac{\partial\phi_1}{\partial f_i}+\Phi_2\frac{\partial\phi_2}{\partial f_i}+\ \ldots\ldots\ldots\ldots+\Phi_m\frac{\partial\phi_m}{\partial f_i}\\&=\lambda\left(\frac{\partial\Pi}{\partial\phi_1}\frac{\partial\phi_1}{\partial f_i}+\frac{\partial\Pi}{\partial\phi_2}\frac{\partial\phi_2}{\partial f_i}+\ldots\ldots+\frac{\partial\Pi}{\partial\phi_m}\frac{\partial\phi_m}{\partial f_i}\right)\\&=-\lambda\frac{\partial\Pi}{\partial f_i}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).\end{aligned}$$

These equations (2), (3), (4) contain the generalisation indicated. They are sufficient to determine the m quantities ϕ, the m quantities Φ, and the (superfluous) quantity λ in terms of the given quantities F and f; and thus it is possible to derive a reduced form $\Sigma\Phi d\phi$ from a given reduced form $\Sigma F df$. The course of the proof shews that, when Π is the most general function possible, the derived reduced form is the most general reduced form possible.

So far as regards the integral equations, which constitute the solution of Pfaff's equation, the most important elements are the quantities ϕ. They are determined by the m equations

$$\Pi = 0,$$

$$\frac{1}{F_1}\frac{\partial\Pi}{\partial f_1} = \frac{1}{F_2}\frac{\partial\Pi}{\partial f_2} = \ldots\ldots = \frac{1}{F_m}\frac{\partial\Pi}{\partial f_m};$$

and therefore they are of the form

$$\phi_i = \psi_i\left(f_1, f_2, \ldots, f_m, \frac{F_1}{F_m}, \frac{F_2}{F_m}, \ldots\ldots, \frac{F_{m-1}}{F_m}\right) \quad \ldots(5),$$

where ψ_i, an arbitrary function, is dependent in form upon Π. There evidently cannot exist any identical relation among these arbitrary functions ψ_i.

It may be noted that, though the functions ψ in the general case are arbitrary and are bound together by no identical relation, yet they are all determined in form by the single arbitrary function Π. It is therefore not justifiable to assume that any m of the $2m-1$ quantities

$$f_1, f_2, \ldots\ldots, f_m, \frac{F_1}{F_m}, \frac{F_2}{F_m}, \ldots\ldots, \frac{F_{m-1}}{F_m}$$

may be chosen to represent m quantities ϕ; any one, or any combination of them, may be chosen for one of the ϕ's, but some of the remaining ϕ's will be, and all of them may be, affected by the form of the function chosen.

Ex. 1. A very simple case, included in (5), is given by an immediate transformation. Since

$$F_1 df_1 + \ldots\ldots + F_m df_m = F_m d\left(f_m + \frac{F_{m-1}}{F_m} f_{m-1} + \ldots\ldots + \frac{F_1}{F_m} f_1\right)$$
$$- F_m f_{m-1}\, d\frac{F_{m-1}}{F_m} - \ldots\ldots - F_m f_1\, d\frac{F_1}{F_m},$$

we have this particular case given by

$$\phi_i = \frac{F_i}{F_m},$$

for $i=1, 2, \ldots\ldots, m-1$; and

$$\phi_m = \frac{F_1}{F_m} f_1 + \frac{F_2}{F_m} f_2 + \ldots\ldots + \frac{F_{m-1}}{F_m} f_{m-1} + f_m.$$

Ex. 2. The equation (2) adopted in the text is the most general form, because the variations of the quantities ϕ and f are thus least limited. The Jacobian, which leads to (2), would equally vanish in virtue of a number of equations of relation between the ϕ's and f's, viz.,

$$\Pi_1=0,\ \Pi_2=0, \ldots\ldots, \Pi_\mu=0;$$

but it is easy to see that the solution to which they lead is only a special case of that which has already been given.

114. The results of the foregoing investigation are: first, *when any solution of Pfaff's equation consisting of the characteristic minimum of integral equations has been obtained, it can be used to obtain the most general solution;* and, second, *the most general solution can be derived from any particular solution.* For, when the quantities $f_1, \ldots, f_m$ are known, the quantities $F_1, F_2, \ldots, F_m$ can be obtained immediately from m independent equations of the form

$$X_r = \sum_{i=1}^{m} F_i \frac{\partial f_i}{\partial x_r};$$

and these are the quantities which are subsidiary to the sought generalisation. It is thus sufficient to have any *particular* solution, in order to obtain the most general solution of Pfaff's differential equation.

115. When Clebsch comes to consider the determination of the elements of any particular solution, it is necessary to discriminate between two cases according as the determinant of the constituents a_{ij} (as they occur in Pfaff's reduction) does not or does vanish. The two cases are the same as occur in that reduction.

First, let that case be considered in which the determinant either does not vanish or vanishes merely owing to conditions among the coefficients which happen to be satisfied; the other case in which the determinant vanishes identically will be subsequently discussed.

We begin with an unconditioned equation containing an even

number ($2n$) of variables; then a reduced form of such an equation

$$\Omega = \sum_{i=1}^{2n} X_i dx_i = 0$$

is

$$F_1 df_1 + \ldots\ldots + F_n df_n = 0.$$

If any expression of the type considered in § 113 be obtained so that we have

$$\phi_n = \phi_n\left(f_1, \ldots, f_n, \frac{F_1}{F_n}, \ldots, \frac{F_{n-1}}{F_n}\right),$$

then by the use of $\phi_n = \text{constant} = a_n$ we have a reduced form, given by

$$\Phi_1\, d\phi_1 + \ldots\ldots + \Phi_{n-1}\, d\phi_{n-1}$$

and associated with $\phi_n = a_n$. By the substitution from $\phi_n = a_n$ for any of the variables, say for x_{2n}, the expression Ω is replaced by another, Ω', containing not more than $2n-1$ variables, say

$$\Omega' = \sum_{i=1}^{2n-1} X_i'\, dx_i,$$

an equivalent reduced form of which is

$$\sum_{j=1}^{n-1} \Phi_j\, d\phi_j,$$

containing only $n-1$ differential elements.

Suppose now that, for $\Omega' = 0$, any base of a differential element of an equivalent reduced form can be obtained in the shape

$$\theta_{n-1} = \theta_{n-1}\left(\phi_1, \phi_2, \ldots, \phi_{n-1}, \frac{\Phi_1}{\Phi_{n-1}}, \ldots, \frac{\Phi_{n-2}}{\Phi_{n-1}}\right),$$

so that a reduced form is given by

$$\Theta_1\, d\theta_1 + \ldots\ldots + \Theta_{n-2} d\theta_{n-2}$$

associated with $\theta_{n-1} = \text{constant} = a_{n-1}$; then by the substitution from $\theta_{n-1} = a_{n-1}$ for any of the variables in Ω', say for x_{2n-1}, the expression Ω' is replaced by another, Ω'', containing not more than $2n-2$ variables, say

$$\Omega'' = \sum_{i=1}^{2n-2} X_i'' dx_i,$$

an equivalent reduced form of which is

$$\sum_{j=1}^{n-2} \Theta_j d\theta_j,$$

containing only $n-2$ differential elements.

Proceeding in this manner we shall ultimately reach an expression $\Omega^{(n-1)}$, containing not more than $n+1$ variables and reducible to a form containing only a single differential element, say to the form

$$\Psi_1 d\psi_1;$$

then an integral system of the original equation is

$$\phi_n = a_n, \ \theta_{n-1} = a_{n-1}, \ \ldots\ldots, \ \psi_1 = a_1.$$

116. This being the general march of Clebsch's derivation of a particular integral system, the first step is the construction of the function ϕ_n. Clebsch's method determines ϕ_n as a solution of a single partial differential equation of the first order; and the form of this equation verifies the inference of § 113 as to the general functional character of ϕ_n. The $(q+1)^{\text{th}}$ step is the construction of the function ξ_{n-q}, which is one of the integrals of the equation $\Omega^{(q)} = 0$ containing $2n-q$ differential elements; it is determined, by Clebsch's method, as a simultaneous solution of $q+1$ partial differential equations all of the first order.

There are therefore, in the first instance, two questions to be considered, similar to one another. The earlier of the two is that in which a differential expression containing $2n$ differential elements has for its reduced form an expression containing n differential elements. The later of the two is that in which a differential expression containing $2m+r$ differential elements is so conditioned that its reduced form contains m differential elements; here the source of the conditions necessary that such a reduction may be possible must be indicated.

Moreover, when any function ϕ_n is adopted for one of the integrals of $\Omega = 0$ and when by means of this integral the number of variables in Ω is reduced by unity, the consequent modification in the form of Ω is partly dependent on the form of ϕ_n; and hence the subsequent integrals may be dependent on this function. In fact, any one of the integrals will in general involve the arbitrary

constants of the other integrals previously obtained: thus we may write

$$\xi_{n-q} = \xi_{n-q}(x, a_n, a_{n-1}, \ldots, a_{n-q+1}).$$

When $a_n, a_{n-1}, \ldots$ are replaced by $\phi_n, \theta_{n-1}, \ldots$ then ξ_{n-q} is a function of the variables x alone; and the form of the function is, in general, affected by the forms of all the integrals which precede it in derivation. It will thus be necessary, as a third question, to examine this effect on the form of such an integral; it will be determined by means of the linear partial differential equations.

These three questions are differently treated by Clebsch. In his first memoir* the first two of them are solved by the construction of the single characteristic differential equation or of the system of such equations corresponding to the two questions indicated; and, for the third of them, the transformation is made by some extremely laboured analysis to the simultaneous equations which determine $\phi, \theta, \ldots, \xi, \ldots, \psi$ as functions of the variables alone. The equations which determine any function ξ have their form affected by the functions similarly determined previous to ξ; and in Clebsch's second memoir† they are obtained directly without the explicit intervention and previous determination of the earlier functions. The discussion of the third question is limited to the most general case of an unconditioned equation in an even number of variables; the extension to a conditioned equation is not given‡.

117. First, then, we have *to obtain the differential equation which is satisfied by the first of the integrals of an unconditioned equation in an even number of variables.* The equation being taken in the form

$$\Omega = \sum_{\mu=1}^{2n} X_\mu \, dx_\mu = 0,$$

* *Crelle*, t. LX. (1862), pp. 193—251.

† *Crelle*, t. LXI. (1863), pp. 146—179.

‡ A large part of these two memoirs of Clebsch, as well as another in *Crelle*, t. LXV. 257—268, is devoted to the theory of the partial differential equations; the work is thus not so entirely limited to the theory of Pfaff's equation as are, for instance, the developments in Natani's process, and much of it really forms an interesting illustration of properties of systems of partial differential equations. For the actual discussion of these classes of equations with the improvements due to Mayer, see §§ 38—41 in Chapter II.

and a reduced form being

$$\Omega = \sum_{\rho=1}^{n} F_\rho df_\rho = 0,$$

we have (for $i = 1, 2, \ldots, 2n$)

$$X_i = F_1 \frac{\partial f_1}{\partial x_i} + F_2 \frac{\partial f_2}{\partial x_i} + \ldots\ldots + F_n \frac{\partial f_n}{\partial x_i},$$

and therefore

$$a_{ij} = \sum_{r=1}^{n} \left(\frac{\partial F_r}{\partial x_j} \frac{\partial f_r}{\partial x_i} - \frac{\partial F_r}{\partial x_i} \frac{\partial f_r}{\partial x_j} \right).$$

Introducing now the quantities y of § 55, we have

$$X_i = \sum_{j=1}^{2n} a_{ij} y_j,$$

and therefore

$$\sum_{r=1}^{n} F_r \frac{\partial f_r}{\partial x_i} = \sum_{j=1}^{2n} y_j \left\{ \sum_{r=1}^{n} \left(\frac{\partial F_r}{\partial x_j} \frac{\partial f_r}{\partial x_i} - \frac{\partial F_r}{\partial x_i} \frac{\partial f_r}{\partial x_j} \right) \right\},$$

this equation holding for $i = 1, 2, 3, \ldots, 2n$. Let

$$\left. \begin{aligned} \sum_{j=1}^{2n} \left(y_j \frac{\partial f_r}{\partial x_j} \right) \qquad\quad &= \theta_r, \\ \sum_{j=1}^{2n} \left(y_j \frac{\partial F_r}{\partial x_j} \right) - F_r &= \Theta_r \end{aligned} \right\} (r = 1, 2, \ldots, n);$$

then the system of equations just obtained is

$$\sum_{r=1}^{n} \left(\Theta_r \frac{\partial f_r}{\partial x_i} - \theta_r \frac{\partial F_r}{\partial x_i} \right) = 0$$

for $i = 1, 2, \ldots, 2n$.

There is thus a system of $2n$ equations, linear and homogeneous in the $2n$ quantities $-\theta$ and Θ. The determinant ∇ of the system is

$$\nabla = \frac{\partial (f_1, \ldots, f_n, F_1, \ldots, F_n)}{\partial (x_1, \ldots\ldots\ldots\ldots\ldots, x_{2n})},$$

and this does not vanish because the $2n$ quantities f and F are independent of one another. Moreover, taking ∇ in the form

$$\nabla = \frac{\partial (F_1, \ldots, F_n, -f_1, \ldots, -f_n)}{\partial (x_1, \ldots\ldots\ldots\ldots\ldots\ldots\ldots, x_{2n})}$$

and multiplying the two values, we find

$$\nabla^2 = \begin{vmatrix} 0\ , & a_{12}, & a_{13}, & \dots \\ a_{21}, & 0\ , & a_{23}, & \dots \\ a_{31}, & a_{32}, & 0\ , & \dots \\ \dots & & & \end{vmatrix};$$

so that ∇ is the Pfaffian $[1, 2, \dots, 2n]$, supposed to be non-evanescent because the equation is unconditioned.

Since the determinant of the system of linear homogeneous equations is not zero, it follows that each of the variables is zero; hence

$$\left.\begin{aligned} \theta_r = 0 = y_1 \frac{\partial f_r}{\partial x_1} + y_2 \frac{\partial f_r}{\partial x_2} + \dots\dots + y_{2n} \frac{\partial f_r}{\partial x_{2n}} \\ \Theta_r = 0 = y_1 \frac{\partial F_r}{\partial x_1} + y_2 \frac{\partial F_r}{\partial x_2} + \dots\dots + y_{2n} \frac{\partial F_r}{\partial x_{2n}} - F_r \end{aligned}\right\} \dots\dots (6),$$

which exist for $r = 1, 2, \dots, n$. And, if, instead of $\sum\limits_{\rho=1}^{n} F_\rho df_\rho$ as a reduced form, $\sum\limits_{\rho=1}^{n} \Phi_\rho d\phi_\rho$ had been taken, the former set of n equations would similarly have been satisfied by any one of the quantities ϕ. Taking then one of them, say ϕ_n, it must satisfy the equation

$$y_1 \frac{\partial \phi}{\partial x_1} + y_2 \frac{\partial \phi}{\partial x_2} + \dots\dots + y_{2n} \frac{\partial \phi}{\partial x_{2n}} = 0 \dots\dots\dots\dots(7).$$

Now of this equation we already have n integrals, viz., $f_1, f_2, \dots, f_n$. But from the second set of n equations we have

$$y_1 \frac{\partial}{\partial x_1}\left(\frac{F_r}{F_n}\right) + y_2 \frac{\partial}{\partial x_2}\left(\frac{F_r}{F_n}\right) + \dots\dots + y_{2n} \frac{\partial}{\partial x_{2n}}\left(\frac{F_r}{F_n}\right) = 0$$

for $r = 1, 2, \dots, n-1$; so that there are $n-1$ other integrals of the partial differential equation given by

$$\frac{F_1}{F_n}, \frac{F_2}{F_n}, \dots\dots, \frac{F_{n-1}}{F_n},$$

and we therefore have $2n-1$ integrals altogether, which are functionally independent. In order to construct the most general solution of (7), only $2n-1$ functionally independent particular solutions are necessary; and the form of this most general solution

is an arbitrary function of the $2n-1$ particular solutions. Hence the general solution of the equation (7) is

$$\phi = \phi_n\left(f_1, \ldots\ldots, f_n, \frac{F_1}{F_n}, \ldots\ldots, \frac{F_{n-1}}{F_n}\right),$$

where ϕ_n denotes any arbitrary function.

This is a verification of the result of § 115: and thus it follows that *a first integral of the given differential equation is furnished by any solution of the equation*

$$y_1\frac{\partial\phi}{\partial x_1} + y_2\frac{\partial\phi}{\partial x_2} + \ldots\ldots + y_{2n}\frac{\partial\phi}{\partial x_{2n}} = 0.$$

It is to be remarked:—first, that every integral of the original differential equation must satisfy this partial equation but that no integral, subsequent to the one initially taken, is completely determined by this equation:—secondly, that we may take any integral of the system

$$\frac{dx_1}{y_1} = \frac{dx_2}{y_2} = \ldots\ldots = \frac{dx_{2n}}{y_{2n}},$$

the subsidiary Pfaffian system, as an integral of the original differential equation, for this system is subsidiary to the complete solution of the partial differential equation:—thirdly, that, even if ∇ vanish (contrary to the initial hypothesis), yet, if not all the Pfaffians of order $2n-2$ vanish, the above partial differential equation (or the subsidiary system) is still valid for the determination of an element ϕ (§ 62) provided we retain the ratios of the vanishing quantities y*.

118. We now pass to the case of a conditioned equation in $(p=)\ 2m+q$ variables

$$\Omega = X_1\,dx_1 + X_2\,dx_2 + \ldots\ldots + X_p\,dx_p = 0,$$

a reduced form of which contains only m differential elements, say

$$\Omega = F_1 df_1 + \ldots\ldots + F_m df_m,$$

and we have *to obtain the differential equations which are satisfied by the first of the integrals of* $\Omega = 0$; as in the general case before

* The only essential difference between this case and the general case is that, in the present case, the fraction F_r/F_n admits of no simplification except the (possible) removal from the numerator and the denominator of a constant factor, while in the general case a variable factor thus disappears. See § 62.

treated, this integral will be an arbitrary function of $f_1, \ldots, f_m$, $\dfrac{F_1}{F_m}, \ldots, \dfrac{F_{m-1}}{F_m}$.

Then

$$X_i = \sum_{r=1}^{m} F_r \frac{\partial f_r}{\partial x_i}$$

and, as before,

$$a_{ij} = \sum_{r=1}^{m} \left(\frac{\partial F_r}{\partial x_j} \frac{\partial f_r}{\partial x_i} - \frac{\partial F_r}{\partial x_i} \frac{\partial f_r}{\partial x_j} \right).$$

Let $\lambda, \mu, \nu, \ldots, \sigma, \ldots, \rho$ be any $2m$ integers of the series $1, 2, \ldots, 2m+q$; and let $q+1$ sets of $2m$ quantities y be introduced, the first set defined by the $2m$ equations

$$X_\theta = \sum_{i=1}^{2m} a_{\theta,i} y_i \qquad (\theta = \lambda, \mu, \nu, \ldots\ldots, \rho),$$

and the remaining sets defined each by $2m$ equations of the form

$$\sum_{i=1}^{2m} a_{\theta,i} \overset{(s)}{y_i} = -a_{\theta,2m+s} \quad (\theta = \lambda, \mu, \nu, \ldots, \rho)$$
$$(s = 1, 2, \ldots, q).$$

Proceeding as in § 117 and using the first set of introduced quantities y, we have

$$\begin{aligned} \sum_{r=1}^{m} F_r \frac{\partial f_r}{\partial x_\theta} &= X_\theta \\ &= \sum_{i=1}^{2m} a_{\theta,i} y_i \\ &= \sum_{i=1}^{2m} y_i \left\{ \sum_{r=1}^{m} \left(\frac{\partial F_r}{\partial x_i} \frac{\partial f_r}{\partial x_\theta} - \frac{\partial F_r}{\partial x_\theta} \frac{\partial f_r}{\partial x_i} \right) \right\} \end{aligned}$$

for each of the values $\lambda, \mu, \nu, \ldots, \rho$ of θ. Let

$$\left. \begin{aligned} \sum_{i=1}^{2m} \left(y_i \frac{\partial f_r}{\partial x_i} \right) \qquad &= \xi_r \\ \sum_{i=1}^{2m} \left(y_i \frac{\partial F_r}{\partial x_i} \right) - F_r &= \Xi_r \end{aligned} \right\} (r = 1, 2, \ldots, m);$$

then the system of equations just obtained is

$$\sum_{r=1}^{m} \left(\Xi_r \frac{\partial f_r}{\partial x_\theta} - \xi_r \frac{\partial F_r}{\partial x_\theta} \right) = 0$$

for the $2m$ values $\theta = \lambda, \mu, \nu, \ldots, \rho$.

It is therefore a system of $2m$ equations linear and homogeneous in the $2m$ variables Ξ_r and $-\xi_r$. As before, the determinant ∇ of the system is a Jacobian of the form

$$\frac{\partial(f_1, \ldots, f_m, F_1, \ldots, F_m)}{\partial(x_\lambda, x_\mu, \ldots\ldots\ldots\ldots\ldots, x_\rho)},$$

and this does not vanish, because the $2m$ quantities f and F are independent of one another; its value is easily proved to be the Pfaffian $[\lambda, \mu, \nu, \ldots, \rho]$ of order $2m$, supposed to be non-evanescent, because m is the smallest number of differential elements in a reduced form of Ω.

Since the determinant of the system of linear equations is not zero, it follows that each of the variables is zero: hence

$$\left.\begin{aligned}\xi_r = 0 &= y_1\frac{\partial f_r}{\partial x_1} + y_2\frac{\partial f_r}{\partial x_2} + \ldots\ldots\ldots\ldots + y_{2m}\frac{\partial f_r}{\partial x_{2m}}\\ \Xi_r = 0 &= y_1\frac{\partial F_r}{\partial x_1} + y_2\frac{\partial F_r}{\partial x_2} + \ldots\ldots\ldots\ldots + y_{2m}\frac{\partial F_r}{\partial x_{2m}} - F_r\end{aligned}\right\} \ldots\ldots (8),$$

which exist for $r = 1, 2, \ldots, m$.

From the second set of equations we have

$$y_1\frac{\partial}{\partial x_1}\left(\frac{F_r}{F_m}\right) + y_2\frac{\partial}{\partial x_2}\left(\frac{F_r}{F_m}\right) + \ldots\ldots + y_{2m}\frac{\partial}{\partial x_{2m}}\left(\frac{F_r}{F_m}\right) = 0,$$

for $r = 1, 2, \ldots, m-1$; and therefore we have $2m-1$ solutions of the equation

$$y_1\frac{\partial\phi}{\partial x_1} + y_2\frac{\partial\phi}{\partial x_2} + \ldots\ldots + y_{2m}\frac{\partial\phi}{\partial x_{2m}} = 0 \;\ldots\ldots\ldots\ldots(9),$$

viz.,

$$f_1, \ldots, f_m, \frac{F_1}{F_m}, \frac{F_2}{F_m}, \ldots, \frac{F_{m-1}}{F_m}.$$

Further, derivatives with regard to $x_{2m+1}, x_{2m+2}, \ldots, x_{2m+q}$ do not occur in this equation; and therefore the most general solution of the equation (9) is an arbitrary function of

$$f_1, f_2, \ldots, f_m, \frac{F_1}{F_m}, \frac{F_2}{F_m}, \ldots, \frac{F_{m-1}}{F_m}, x_{2m+1}, x_{2m+2}, \ldots, x_{2m+q}.$$

Using now any other set of the subsidiary quantities y, say $y_1^{(s)}, \ldots\ldots, y_{2m}^{(s)}$, we have, for $s = 1, 2, \ldots, q$,

$$-\sum_{r=1}^{m}\left(\frac{\partial F_r}{\partial x_{2m+s}}\frac{\partial f_r}{\partial x_\theta}-\frac{\partial F_r}{\partial x_\theta}\frac{\partial f_r}{\partial x_{2m+s}}\right)=-a_{\theta,2m+s}$$

$$=\sum_{i=1}^{2m} a_{\theta,i}\,y_i^{(s)}$$

$$=\sum_{i=1}^{2m} y_i^{(s)}\left\{\sum_{r=1}^{m}\left(\frac{\partial F_r}{\partial x_i}\frac{\partial f_r}{\partial x_\theta}-\frac{\partial F_r}{\partial x_\theta}\frac{\partial f_r}{\partial x_i}\right)\right\}$$

for each of the $2m$ values $\lambda, \mu, \nu, \ldots, \rho$ of θ. Let

$$\left.\begin{aligned}\sum_{i=1}^{2m}\left(y_i^{(s)}\frac{\partial f_r}{\partial x_i}\right)+\frac{\partial f_r}{\partial x_{2m+s}}&=\xi_r^{(s)}\\ \sum_{i=1}^{2m}\left(y_i^{(s)}\frac{\partial F_r}{\partial x_i}\right)+\frac{\partial F_r}{\partial x_{2m+s}}&=\Xi_r^{(s)}\end{aligned}\right\}(r=1,2,\ldots,m);$$

then the system of equations just obtained is

$$\sum_{r=1}^{m}\left(\Xi_r^{(s)}\frac{\partial f_r}{\partial x_\theta}-\xi_r^{(s)}\frac{\partial F_r}{\partial x_\theta}\right)=0$$

for the $2m$ values $\lambda, \mu, \nu, \ldots, \rho$ of θ. The determinant of this system of $2m$ equations, linearly homogeneous in $2m$ variables, does not vanish; and therefore each of the variables vanishes, so that

$$\left.\begin{aligned}\xi_r^{(s)}&=0=y_1^{(s)}\frac{\partial f_r}{\partial x_1}+y_2^{(s)}\frac{\partial f_r}{\partial x_2}+\ldots\ldots+y_{2m}^{(s)}\frac{\partial f_r}{\partial x_{2m}}+\frac{\partial f_r}{\partial x_{2m+s}}\\ \Xi_r^{(s)}&=0=y_1^{(s)}\frac{\partial F_r}{\partial x_1}+y_2^{(s)}\frac{\partial F_r}{\partial x_2}+\ldots\ldots+y_{2m}^{(s)}\frac{\partial F_r}{\partial x_{2m}}+\frac{\partial F_r}{\partial x_{2m+s}}\end{aligned}\right\}\ldots(10),$$

which exist for $r=1, 2, \ldots, m$ and for $s=1, 2, \ldots, q$.

Hence we have, as solutions of the equation

$$y_1^{(s)}\frac{\partial\phi}{\partial x_1}+y_2^{(s)}\frac{\partial\phi}{\partial x_2}+\ldots\ldots+y_{2m}^{(s)}\frac{\partial\phi}{\partial x_{2m}}+\frac{\partial\phi}{\partial x_{2m+s}}=0\ldots\ldots(11),$$

the $2m$ quantities $f_1, \ldots, f_m, F_1, \ldots, F_m$; and, since derivatives with regard to $x_{2m+1}, x_{2m+2}, \ldots, x_{2m+s-1}, x_{2m+s+1}, \ldots, x_{2m+q}$ do not occur in (11), the most general solution of this equation is an arbitrary function of

$$f_1, \ldots, f_m, F_1, \ldots, F_m, x_{2m+1}, \ldots, x_{2m+s-1}, x_{2m+s+1}, \ldots, x_{2m+q}.$$

Consider now the aggregate of equations made up of the q equations represented by (11) and of the single equation (9), viz.,

$$\left.\begin{aligned}
A\phi &= y_1\frac{\partial\phi}{\partial x_1} + y_2\frac{\partial\phi}{\partial x_2} + \dots\dots\dots + y_{2m}\frac{\partial\phi}{\partial x_{2m}} &&= 0\\
A_1\phi &= y_1^{(1)}\frac{\partial\phi}{\partial x_1} + y_2^{(1)}\frac{\partial\phi}{\partial x_2} + \dots\dots\dots + y_{2m}^{(1)}\frac{\partial\phi}{\partial x_{2m}} + \frac{\partial\phi}{\partial x_{2m+1}} &&= 0\\
A_2\phi &= y_1^{(2)}\frac{\partial\phi}{\partial x_1} + y_2^{(2)}\frac{\partial\phi}{\partial x_2} + \dots\dots\dots + y_{2m}^{(2)}\frac{\partial\phi}{\partial x_{2m}} + \frac{\partial\phi}{\partial x_{2m+2}} &&= 0\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
A_q\phi &= y_1^{(q)}\frac{\partial\phi}{\partial x_1} + y_2^{(q)}\frac{\partial\phi}{\partial x_2} + \dots\dots\dots + y_{2m}^{(q)}\frac{\partial\phi}{\partial x_{2m}} + \frac{\partial\phi}{\partial x_{2m+q}} &&= 0
\end{aligned}\right\}\dots(12),$$

being $q+1$ in all. It is an immediate inference from the character of the individual general solutions of each of the equations that the most general simultaneous solution of the system of equations (12) treated as simultaneous is an arbitrary function of

$$f_1, f_2, \dots, f_m, \frac{F_1}{F_m}, \frac{F_2}{F_m}, \dots, \frac{F_{m-1}}{F_m}.$$

If, instead of beginning with a reduced form $\Sigma F df$ as equivalent to Ω, we had begun with an equivalent reduced form $\Sigma\Phi d\phi$, the preceding equations would have been satisfied by the elements ϕ: and therefore the most general element entering into a reduced form equivalent to Ω is given by

$$\phi_m = \phi_m\left(f_1, f_2, \dots, f_m, \frac{F_1}{F_m}, \frac{F_2}{F_m}, \dots, \frac{F_{m-1}}{F_m}\right),$$

where ϕ_m denotes any arbitrary function.

This again is a verification of the result of § 115; and it follows that *a first integral of the given conditioned differential equation is furnished by any simultaneous solution of the system* (12) *of simultaneous equations.*

119. Several remarks are to be made at this point.

(i). The form of the characteristic equations determining the elements of a reduced form must be independent of the choice of the $2m$ terms λ, μ, ..., ρ from the series 1, 2, ..., $2m+q$; and therefore the values of the coefficients must be the same whatever be the selection thus made. The conditions, necessary that this may hold, are the conditions under which the equation involving $2m+q$ variables can be satisfied by only m integral equations; and

they can, without difficulty, be expressed in the forms already given in the discussion of Natani's method (§ 100). Moreover, as these conditions are explicitly known, they are sufficient to indicate the number of equations in the integral system by which the given differential equation is satisfied.

And as all Pfaffians of order $2m$ will usually not vanish, m being the number of differential elements in a reduced form, we should in the first instance choose $\lambda, \mu, \ldots, \rho$ so as to give a non-evanescent Pfaffian $[\lambda, \mu, \ldots, \rho]$, if it should happen that some of the Pfaffians of this order $2m$ vanish. The first set of subsidiary quantities y is then similar to the set in § 117; each of the remaining sets is easily seen to be made up of quotients of Pfaffians of order $2m$ by the Pfaffian $[\lambda, \mu, \ldots, \rho]$. For instance, if

$$\sum_{i=1}^{6} X_i dx_i = 0$$

have a reduced form containing only two differential elements and if we denote $X_1[23] + X_2[31] + X_3[12]$ by $[0123]$, then the three equations which determine the first integral are easily seen to be

$$\left.\begin{array}{l} [2340]\dfrac{\partial\phi}{\partial x_1} + [3401]\dfrac{\partial\phi}{\partial x_2} + [4012]\dfrac{\partial\phi}{\partial x_3} + [0123]\dfrac{\partial\phi}{\partial x_4} = 0 \\ [2345]\dfrac{\partial\phi}{\partial x_1} + [3451]\dfrac{\partial\phi}{\partial x_2} + [4512]\dfrac{\partial\phi}{\partial x_3} + [5123]\dfrac{\partial\phi}{\partial x_4} + [1234]\dfrac{\partial\phi}{\partial x_5} = 0 \\ [2346]\dfrac{\partial\phi}{\partial x_1} + [3461]\dfrac{\partial\phi}{\partial x_2} + [4612]\dfrac{\partial\phi}{\partial x_3} + [6123]\dfrac{\partial\phi}{\partial x_4} + [1234]\dfrac{\partial\phi}{\partial x_6} = 0 \end{array}\right\}.$$

(ii). The following is the march of the general reduction of an equation $\Omega = 0$: the function ϕ, when determined as a simultaneous solution of the system (12), is used to remove one of the variables from the equation, by taking

$$\phi = \text{constant};$$

we then have a differential equation with one variable fewer and integrable by one equation fewer, that is, we have the next simpler form of the equation already treated.

(iii). Every integral of the original differential equation must satisfy the system (12) of characteristic simultaneous partial differential equations; but no integral, subsequent to the one initially taken, is completely determined by the system (12). In fact, for each new integral to be determined, the new characteristic system, corresponding to (12), must be constructed.

Note. The solution of the equations (12) can be effected by the method of Jacobi. They form a complete system: it is easy to verify that the Jacobian conditions, which may be expressed in the form

$$A\,A_i\phi - A_i A\,\phi = 0,$$
$$A_j A_i\phi - A_i A_j\phi = 0,$$

are all satisfied.

The subsidiary system of $A\phi=0$ leads to $2m-1$ subsidiary equations of the first order or to an ordinary differential equation of the $(2m-1)^{\text{th}}$ order. When the general form of ϕ satisfying $A\phi=0$ is obtained and is substituted in $A_1\phi=0$, the latter changes into a new equation, of which the independent variables may be made $f_1, \ldots\ldots, f_m, \frac{F_1}{F_m}, \ldots\ldots, \frac{F_{m-1}}{F_m}$ and x_{2m+1}; so that the complete solution of $A_1\phi=0$ may be made to depend upon the integration of an ordinary differential equation of the $(2m-1)^{\text{th}}$ order. And so on for each of the equations in the system (12); so that the complete solution of that system may require the integration of $q+1$ equations, each of order $2m-1$.

Hence this method would require the solution of

(i) one ordinary differential equation of order $2n-1$,
(ii) two . . . . equations . . . $2n-3$,
(iii) three . . . . $2n-5$,
....
(n) n 1,

for the complete determination of the general integral system of an unconditioned differential equation in $2n$ variables; and each of these solutions gives one of the integrals in the integral system.

The similar results for a conditioned equation are easily inferred.

120. The foregoing constitutes what may be called Clebsch's First Method, as applied either to an unconditioned equation in an even number of variables or to a conditioned equation. It requires the transformation of the differential equation after each integral has been obtained, in order to construct the system of characteristic equations which shall determine the next succeeding integral.

In order to obviate some of this labour Clebsch in his first memoir transforms, as already (§ 116) stated, these characteristic equations, so that there occur in them only the coefficients of the original differential equation and the integrals already obtained and not the coefficients of the differential equation modified by each successive integral. In his second memoir he obtains directly the same results as are obtained by these transformations: to this direct investigation, which may be called Clebsch's Second Method, we now proceed.

121. Clebsch's second method applies solely to the case of an unconditioned equation in an even number of variables, so that a reduced form of

$$\sum_{i=1}^{2n} X_i dx_i$$

(with none of the critical conditions among the coefficients satisfied) is

$$\sum_{r=1}^{n} F_r df_r.$$

Then the $2n$ quantities F and f are $2n$ independent functions of the $2n$ independent variables x; and therefore, conversely, the $2n$ quantities x are $2n$ independent functions of the quantities F and f, which may thus if convenient be considered as a complete set of independent variables.

The object of the method is to obtain the characteristic equations which determine the n quantities f without regard to the effect on the original equation of the use of successive integrals: and these equations are immediately derived by means of the following lemma :—

Let P denote the non-evanescent Pfaffian $[1, 2, \ldots, 2n]$ and let R_{ij} be the coefficient of a_{ij} in P so that

$$R_{ij} = \frac{\partial P}{\partial a_{ij}},$$

and R_{ij} differs only by the factor $(-1)^{1+i-j}$ from P_{ij} (§ 59): then, if ϕ and ψ be any functions of the variables x,

$$\left.\begin{aligned} \sum_{r=1}^{n} F_r \frac{\partial \phi}{\partial F_r} = \quad (\phi) \quad &= \sum_{i=1}^{2n} \sum_{j=1}^{2n} \frac{R_{ij}}{P} X_i \frac{\partial \phi}{\partial x_j} \\ \sum_{r=1}^{n} \left(\frac{\partial \psi}{\partial F_r} \frac{\partial \phi}{\partial f_r} - \frac{\partial \phi}{\partial F_r} \frac{\partial \psi}{\partial f_r} \right) = [\phi, \psi] &= \sum_{i=1}^{2n} \sum_{j=1}^{2n} \frac{R_{ij}}{P} \frac{\partial \phi}{\partial x_i} \frac{\partial \psi}{\partial x_j} \end{aligned}\right\},$$

which evidently are propositions relating to transformation of independent variables.

We have, as before,

$$a_{ij} = \sum_{r=1}^{n} \left(\frac{\partial F_r}{\partial x_j} \frac{\partial f_r}{\partial x_i} - \frac{\partial F_r}{\partial x_i} \frac{\partial f_r}{\partial x_j} \right).$$

Let c_{ij} denote the corresponding quantity derived when the variables x are looked upon as functions of F and f, viz.,

$$c_{ij} = \sum_{r=1}^{n} \left(\frac{\partial x_j}{\partial F_r} \frac{\partial x_i}{\partial f_r} - \frac{\partial x_i}{\partial F_r} \frac{\partial x_j}{\partial f_r} \right).$$

Then, if λ and μ denote any two of the quantities f and F, the value of

$$\sum_{i=1}^{2n} \frac{\partial \lambda}{\partial x_i} \frac{\partial x_i}{\partial \mu}$$

is zero, if λ and μ be different quantities, and is unity, if they be the same quantities. In virtue of this relation it is easy to shew that

$$\sum_{j=1}^{2n} c_{ij}\, a_{sj} \qquad (s = 1, 2, \ldots, 2n)$$

is zero, if i and s be different, and is equal to 1, if i and s be the same; and therefore solving the $2n$ consequent equations we have

$$\begin{aligned} P^2 c_{ij} &= \text{minor of } a_{ij} \text{ in } P^2 \\ &= P \frac{\partial P}{\partial a_{ij}}, \end{aligned}$$

and therefore

$$c_{ij} = \frac{1}{P} \frac{\partial P}{\partial a_{ij}} = \frac{R_{ij}}{P} *.$$

Hence for the first of the two transformations we have

$$\begin{aligned} (\phi) &= \sum_{i=1}^{2n} \sum_{j=1}^{2n} \frac{R_{ij}}{P} X_i \frac{\partial \phi}{\partial x_j} \\ &= \sum_{i=1}^{2n} \sum_{j=1}^{2n} c_{ij} \frac{\partial \phi}{\partial x_j} \sum_{r=1}^{n} F_r \frac{\partial f_r}{\partial x_i} \\ &= \sum_{i=1}^{2n} \sum_{j=1}^{2n} \sum_{r=1}^{n} \sum_{s=1}^{n} \left\{ F_r \frac{\partial f_r}{\partial x_i} \frac{\partial \phi}{\partial x_j} \left(\frac{\partial x_j}{\partial F_s} \frac{\partial x_i}{\partial f_s} - \frac{\partial x_i}{\partial F_s} \frac{\partial x_j}{\partial f_s} \right) \right\}. \end{aligned}$$

Taking now the terms separately, the typical expression of the first is

$$F_r \frac{\partial f_r}{\partial x_i} \frac{\partial x_i}{\partial f_s} \frac{\partial \phi}{\partial x_j} \frac{\partial x_j}{\partial F_s}.$$

* This theorem is given by Brioschi, *La teorica dei determinanti* (1854), p. 60: the equations from which the result is obtained are due to Cauchy (l. c.).

The sum of all these which have the same r, s, and j is

$$F_r \frac{\partial \phi}{\partial x_j} \frac{\partial x_j}{\partial F_s} \sum_{i=1}^{2n} \frac{\partial f_r}{\partial x_i} \frac{\partial x_i}{\partial f_s},$$

which is zero unless s be the same as r: and so the sum is

$$F_r \frac{\partial \phi}{\partial x_j} \frac{\partial x_j}{\partial F_r}.$$

The sum of all these which have the same r is

$$F_r \sum_{j=1}^{2n} \frac{\partial \phi}{\partial x_j} \frac{\partial x_j}{\partial F_r} = F_r \frac{\partial \phi}{\partial F_r}.$$

Hence the sum of all the first terms is

$$\sum_{r=1}^{n} F_r \frac{\partial \phi}{\partial F_r}.$$

The typical expression of the second term is

$$F_r \frac{\partial \phi}{\partial x_j} \frac{\partial x_j}{\partial f_s} \frac{\partial f_r}{\partial x_i} \frac{\partial x_i}{\partial F_s}:$$

the sum of all these which have the same r, s and j is

$$F_r \frac{\partial \phi}{\partial x_j} \frac{\partial x_j}{\partial f_s} \sum_{i=1}^{2n} \frac{\partial f_r}{\partial x_i} \frac{\partial x_i}{\partial F_s},$$

which is always zero. Hence we have

$$(\phi) = \sum_{r=1}^{n} F_r \frac{\partial \phi}{\partial F_r},$$

thus proving the first of the propositions.

For the second of them, we have

$$\begin{aligned}
[\phi, \psi] &= \sum_{i=1}^{2n} \sum_{j=1}^{2n} \frac{R_{ij}}{P} \frac{\partial \phi}{\partial x_i} \frac{\partial \psi}{\partial x_j} \\
&= \sum_{i=1}^{2n} \sum_{j=1}^{2n} c_{ij} \frac{\partial \phi}{\partial x_i} \frac{\partial \psi}{\partial x_j} \\
&= \sum_{i=1}^{2n} \sum_{j=1}^{2n} \sum_{r=1}^{n} \left\{ \frac{\partial \phi}{\partial x_i} \frac{\partial \psi}{\partial x_j} \left(\frac{\partial x_j}{\partial F_r} \frac{\partial x_i}{\partial f_r} - \frac{\partial x_i}{\partial F_r} \frac{\partial x_j}{\partial f_r} \right) \right\} \\
&= \sum_{r=1}^{n} \left(\frac{\partial \psi}{\partial F_r} \frac{\partial \phi}{\partial f_r} - \frac{\partial \phi}{\partial F_r} \frac{\partial \psi}{\partial f_r} \right),
\end{aligned}$$

which proves the second of the propositions.

122. The derivation of the desired characteristic equations can now be effected. In the propositions just proved, no restrictions were laid upon the functions ϕ and ψ: and thus, particular cases can be obtained by equating them to some of the functions F and f.

We have from the first proposition

$$(f_i) = \sum_{r=1}^{n} F_r \frac{\partial f_i}{\partial F_r} = 0;$$

$$(F_i) = F_i,$$

and therefore

$$(F_i \div F_n) = 0,$$

for all the values of $i = 1, 2, \ldots, n$.

We have from the second

$$[f_i, f_j] = 0$$

for all combinations of i and j from the series $1, 2, \ldots, n$; also

$$[F_i, F_j] = 0;$$

and

$$[f_i, F_j] = 0,$$

if i and j be different, while

$$[f_i, F_i] = 1$$

for each of the n values of i. And we further derive

$$\left[\frac{F_i}{F_n}, \frac{F_j}{F_n}\right] = 0 = \left[f_i, \frac{F_j}{F_n}\right],$$

if i and j be different, and

$$0 = \left[f_i, \frac{F_i}{F_n}\right],$$

for each of the n values of i.

The only equations, into which the quantities f alone enter, are the n equations

$$(f_i) = 0$$

and the $\frac{1}{2}n(n-1)$ equations

$$[f_i, f_j] = 0;$$

and therefore we have the result:—

The n simultaneous integrals of the unconditioned differential equation

$$\sum_{i=1}^{2n} X_i dx_i = 0$$

$$\left.\begin{array}{l}\textit{satisfy the } n \textit{ partial differential equations}\\ \qquad (f_m)=\sum\limits_{i=1}^{2n}\sum\limits_{j=1}^{2n}\dfrac{R_{ij}}{P}X_i\dfrac{\partial f_m}{\partial x_j}=0\quad (m=1,2,\ldots,n)\\ \textit{and the } \tfrac{1}{2}n(n-1) \textit{ partial differential equations}\\ \qquad [f_l, f_m]=\sum\limits_{i=1}^{2n}\sum\limits_{j=1}^{2n}\dfrac{R_{ij}}{P}\dfrac{\partial f_l}{\partial x_i}\dfrac{\partial f_m}{\partial x_j}=0\quad (l, m=1,2,\ldots,n)\end{array}\right\}\ldots(13).$$

Further, the other obtained relations shew that these equations are satisfied when any of the quantities f_m is replaced by any one of the quantities F_r/F_n; and therefore we infer, in connection with the argument of § 113, that *these differential equations are sufficient as well as necessary for the determination of the functions f.*

123. By referring to § 59, it will be seen that

$$\sum_{i=1}^{2n}\frac{R_{ij}}{P}X_i$$

is equal to $-y_j$: and therefore the equation $(\phi)=0$ is the same as

$$\sum_{j=1}^{2n} y_j\frac{\partial\phi}{\partial x_j}=0,$$

viz., is the same as the equation (7) which occurs in Clebsch's first method.

Further, it may be noticed (though it is not remarked by Clebsch) that the factor $\frac{1}{P}$ may be *removed* from all the equations $(\phi)=0$, $[\phi, \psi]=0$ *even in the case when P vanishes,* provided that all the Pfaffians of order $2n-2$ do not vanish and no one of the characteristic equations become illusory at any stage. This statement may be justified by the method adopted, for the similar question, in § 62*. Hence *Clebsch's second method applies to an equation in 2n variables even when the Pfaffian of all the coefficients vanishes, provided the Pfaffians of next lower order 2n − 2 do not all vanish and no characteristic equation become illusory at any stage.*

* But the method will not apply in the case of an equation in an odd number of variables; for no modification of the coefficients can prove effective, since the Pfaffian of odd order is necessarily evanescent. Hence the present form of the method is not applicable to an equation in an odd number of variables; and extensions of this method to such an equation, as well as to conditioned equations in an even number of variables, were not made by Clebsch.

124. The solution, then, of the original differential equation $\Omega = 0$ depends, by this method, on the integration of a system (13) of simultaneous linear partial differential equations of the first order. To determine the first member of the integral system which constitutes the required solution, say f_1, we take any integral of the equation

$$(\phi) = 0$$

or of the corresponding associated subsidiary system.

To determine the second member of the solution, say f_2, we take a simultaneous integral, other than f_1, of the two equations

$$(\phi) = 0, \quad [f_1, \phi] = 0.$$

To determine the third member of the solution, say f_3, we take a simultaneous integral, other than f_1 or f_2 or any function of them, of the three equations

$$(\phi) = 0, \quad [f_1, \phi] = 0, \quad [f_2, \phi] = 0.$$

And so on.

It is evident that each of the members thus obtained is a function of the variables only; that the form of every member is affected by all the previously obtained members of the desired integral system; and that the way, in which the effect of these previously obtained members is brought about, is indicated by the form of the determining system of simultaneous equations. Thus the second and the third questions of § 116 are solved.

125. The solution of the system of simultaneous partial differential equations which determine any member of the integral system can be effected in accordance with Mayer's theory of such systems*. We shall not enter here into any details of such solution, as the developments belong less to the theory of Pfaff's equation than to the theory of partial differential equations. It may, however, be convenient to state here two of the results appertaining to the present systems; the proof of these results is not difficult.

First, whatever be the values of any three functions ϕ, ψ, χ of the variables, the relations

$$[[\phi, \psi], \chi] + [[\psi, \chi], \phi] + [[\chi, \phi], \psi] = 0$$

and

$$[(\phi), \psi] + [\phi, (\psi)] + [\psi, \phi] = ([\phi, \psi])$$

are identically satisfied: the simplest proof of these relations is obtained by considering all the quantities as functions of F and f. Now in the deter-

* See note to § 116, p. 200.

mination of each of the members of the integral system, it is in every case necessary to begin with a functionally new solution of $(\phi)=0$; and for this purpose the following results, easily derivable from the preceding identical relations, are useful:

(i) If ϕ and ψ be two solutions, functionally distinct, of $(\phi)=0$, then

$$\frac{[[\phi, \psi], \phi]}{[\phi, \psi]^2} \text{ and } \frac{[[\phi, \psi], \psi]}{[\phi, \psi]^2}$$

are two other solutions: they may, however, be illusory, or they may be expressible in terms of ϕ and ψ.

(ii) If ϕ, ψ, χ be three solutions, functionally distinct, of $(\phi)=0$, then each of the quantities

$$\frac{[\phi, \psi]}{[\psi, \chi]}, \frac{[\psi, \chi]}{[\chi, \phi]}, \frac{[\chi, \phi]}{[\phi, \psi]}$$

is also a solution; so that, if they be not illusory and be functionally distinct from combinations of ϕ, ψ and χ, two other new solutions are given.

(iii) If in the equation $[\phi, f]=0$ the unknown quantity be ϕ and if f be considered known, then from two solutions ψ and χ a third is given in the form

$$[\psi, \chi],$$

if it be not illusory and it be functionally distinct from combinations of ψ and χ.

Second, the system of simultaneous equations which determine f_i is

$$(\phi)=0,\ [f_1, \phi]=0,\ [f_2, \phi]=0, \ldots\ldots,\ [f_{i-1}, \phi]=0;$$

Clebsch replaces these equations by a linear combination of them, constituting an equivalent system and constructed as follows. A set of functions Ω_1, $\Omega_2, \ldots\ldots, \Omega_i$ being introduced, the equations

$$\left.\begin{aligned}(\phi)&=(\Omega_1)(\phi)_1 + (\Omega_2)(\phi)_2+\ldots\ldots+ (\Omega_i)(\phi)_i\\ [f_r, \phi]&=[f_r, \Omega_1](\phi)_1+[f_r, \Omega_2](\phi)_2+\ldots\ldots+[f_r, \Omega_i](\phi)_i\end{aligned}\right\}$$

(where $r=1, 2, \ldots\ldots, i-1$) are formed; and the sole limitation on the functions Ω, which can otherwise be chosen at will, is that the determinant of the right-hand side does not vanish. Then the former system can be replaced by

$$(\phi)_1=0,\ (\phi)_2=0, \ldots\ldots,\ (\phi)_i=0;$$

the new system satisfies the relation

$$((\phi)_\lambda)_\mu=((\phi)_\mu)_\lambda$$

and thus is a complete Jacobian system (§ 38).

Ex. As an illustration of Clebsch's methods, consider the equation

$$x_4dx_1+2x_1dx_2+x_2dx_3+2x_3dx_4=0.$$

We have

$$R_{34}=[12]=-2,\quad R_{42}=[13]=0,\quad R_{23}=[14]=\ \ 1,$$
$$R_{12}=[34]=-2,\quad R_{13}=[42]=0,\quad R_{14}=[23]=-1,$$

and
$$[1234]=3,$$

so that the Pfaffian does not vanish, and may be omitted from the equations. The first of these characteristic equations $(\phi)=0$ is

$$(\phi)=(4x_1+2x_3)\frac{\partial\phi}{\partial x_1}-(2x_4+x_2)\frac{\partial\phi}{\partial x_2}+(2x_1+4x_3)\frac{\partial\phi}{\partial x_3}-(x_4+2x_2)\frac{\partial\phi}{\partial x_4}=0.$$

Subsidiary equations are

$$\frac{dx_1}{4x_1+2x_3}=\frac{-dx_2}{2x_4+x_2}=\frac{dx_3}{2x_1+4x_3}=\frac{-dx_4}{x_4+2x_2}=d\theta;$$

and the necessary three independent special integrals are obtainable by the elimination of θ between

$$x_1+x_3=Ae^{6\theta},\qquad x_2-x_4=De^{\theta},$$
$$x_1-x_3=Be^{2\theta},\qquad x_2+x_4=Ce^{-3\theta}.$$

We may therefore take

$$f_2=\psi=(x_2+x_4)(x_2-x_4)^3,$$

as an integral of $(\phi)=0$ and consequently as a first integral of the original equation.

Clebsch's first method would require the elimination of say x_4 and dx_4 from the original differential equation by means of its first integral $f_2=a$, and would then leave an integrable expression. Instead of using this method, we shall adopt the second method, which requires for the deduction of the second integral a simultaneous solution other than ψ of

$$(\phi)=0,\qquad [\phi,\psi]=0.$$

Taking the second equation, substituting for ψ and simplifying by the rejection of negligible factors, we find that it takes the form

$$x_2\frac{\partial\phi}{\partial x_1}-x_4\frac{\partial\phi}{\partial x_3}=0.$$

We must now find some common solution of this and $(\phi)=0$ which is distinct from ψ.

We proceed as follows: The subsidiary equations of the new partial equation are

$$\frac{dx_1}{x_2}=\frac{dx_2}{0}=\frac{dx_3}{-x_4}=\frac{dx_4}{0},$$

three independent integrals of which are x_2, x_4, $(u=)\,x_1x_4+x_2x_3$; hence the most general solution of the new equation is

$$\phi=f(x_2,x_4,u),$$

where f implies any functional combination whatever. It is now necessary

to find what forms of f will satisfy $(\phi)=0$; on substituting, we have after a little reduction

$$(\phi)=3u\frac{\partial f}{\partial u}-(2x_4+x_2)\frac{\partial f}{\partial x_2}-(x_4+2x_2)\frac{\partial f}{\partial x_4}=0,$$

an equation in which all the coefficients are expressible in terms of x_2, x_4 and u. (This result is to be expected because $(\phi)=0$ and $[\phi, \psi]=0$ satisfy the Jacobian conditions.) The subsidiary equations for the new form of $(\phi)=0$ are

$$\begin{aligned}\frac{du}{-3u}=\frac{dx_2}{2x_4+x_2}&=\frac{dx_4}{x_4+2x_2}\\ &=\frac{dx_2+dx_4}{3(x_2+x_4)}=\frac{dx_2-dx_4}{-(x_2-x_4)},\end{aligned}$$

two independent integrals of which are

$$u(x_2+x_4),\quad (x_2+x_4)(x_2-x_4)^3.$$

The second of these is ψ, which may be expected to occur as a common solution of $(\phi)=0$, $[\phi, \psi]=0$; hence ϕ is merely a function of $(x_2+x_4)u$, and thus we may take as a system of two integrals of the given differential equation

$$\left.\begin{aligned}f_1&=(x_2+x_4)(x_1x_4+x_2x_3)\\ f_2&=(x_2+x_4)(x_2-x_4)^3\end{aligned}\right\}.$$

The corresponding values of F_1 and F_2 are

$$F_1=\frac{1}{x_2+x_4},\qquad F_2=\frac{x_1-x_3}{(x_2+x_4)(x_2-x_4)^2}.$$

When we take as the first integral

$$g_2=\psi=(x_1+x_3)(x_1-x_3)^{-3}$$

and proceed similarly, we find

$$g_1=(x_1+x_3)^{-1}(x_1x_2+x_3x_4)^2,$$

which thus give another system of solutions. It is easy to shew, in verification of § 113, that

$$\left.\begin{aligned}g_2&=\frac{\lambda f_2+2f_1}{\lambda^3 f_2^2}\\ g_1&=\frac{(\lambda f_2+f_1)^2}{\lambda f_2+2f_1}\end{aligned}\right\}\text{ where }\lambda=\frac{F_2}{F_1}.$$

When we take as the first integral

$$h_2=\psi=(x_1+x_3)(x_2+x_4)^2$$

and proceed similarly, we find

$$h_1=(x_2-x_4)^2(x_1-x_3)^{-1},$$

thus giving another system of solutions: and, for the similar verification,

$$h_2=\lambda f_2+2f_1,\qquad h_1\lambda=1.$$

126. We now pass to the consideration of the equations which cannot be treated by either of the foregoing methods. The case which most frequently arises is that of an unconditioned equation in an odd number of variables, the Pfaffian of which necessarily vanishes; other cases are the exceptions of § 119 in which all the Pfaffians of a given order vanish, though not those of the next lower order.

Let it be supposed that the equation

$$\Omega = \sum_{i=1}^{p} X_i dx_i = 0$$

has a reduced form

$$Fdf + F_1 df_1 + \ldots + F_m df_m,$$

and that no equivalent form can be obtained in a smaller number of differential elements than $m+1$; then the most frequent case is that in which $p = 2m+1$, and for all other cases $p \mathrel{\overline{\overline{>}}} 2(m+1)$. And, in all cases, the Pfaffian of the original equation $\Omega = 0$ vanishes.

First, if $p = 2m+1$, then there are $2m+2$ functions f and F of only $2m+1$ variables; hence there must be one relation among these functions and, in the case of an unconditioned equation, not more than one relation. But if any one of the coefficients, say F, be a constant, which may be taken to be unity, then there is no such relation among the m coefficients $F_1, \ldots, F_m$ and the $m+1$ bases of differential elements $f, f_1, \ldots, f_m$.

In all the other cases, all Pfaffians of order $2m+2$ vanish. Now the square of the determinant

$$\frac{\partial(f, f_1, \ldots, f_m, F, F_1, \ldots, F_m)}{\partial(x_\alpha, x_\beta, \ldots\ldots\ldots\ldots\ldots\ldots\ldots, x_\tau)},$$

where $\alpha, \beta, \ldots, \tau$ are any $2m+2$ integers from the series $1, 2, \ldots, p$, is the square of the Pfaffian $[\alpha, \beta, \ldots, \tau]$, which, being of order $2m+2$, vanishes by hypothesis; and therefore each of the foregoing determinants, for each selection of $2m+2$ integers, vanishes. Hence one functional relation must exist among the quantities $f, f_1, \ldots, f_m, F, F_1, \ldots, F_m$. But, also by hypothesis, all the Pfaffians of order $2m$ do not vanish; and therefore all the first minors of the preceding determinant may not vanish; thus there are not two

(or more) relations among the quantities f and F*, that is, there is only a single functional relation.

If however any one of the coefficients, say F, be unity, the condition that the determinant shall vanish is satisfied: and there is then no relation.

Hence, in every case, the reduced form is either of the type

$$\sum_{i=0}^{m} F_i df_i$$

with a single identical relation among the quantities f and F, or it is of the type

$$df + \sum_{i=1}^{m} F_i df_i.$$

127. When in any manner a particular reduced form has been obtained, it can be generalised as follows. Let such a form be

$$Fdf + F_1 df_1 + \ldots\ldots + F_m df_m$$

with a single functional relation, say

$$F = \Psi(f, f_1, \ldots, f_m, F_1, \ldots, F_m),$$

among the quantities; the alternative form will be found to arise in the course of the generalisation. Let the most general reduced form be

$$\Phi d\phi + \Phi_1 d\phi_1 + \ldots\ldots + \Phi_m d\phi_m,$$

with the existence of a corresponding functional relation. Then taking $f, f_1, \ldots, f_m, F_1, \ldots, F_m, x_{2m+2}, \ldots, x_p$ as independent variables, we have

$$\left.\begin{aligned} F &= \Phi \frac{\partial \phi}{\partial f} + \Phi_1 \frac{\partial \phi_1}{\partial f} + \ldots\ldots + \Phi_m \frac{\partial \phi_m}{\partial f} \\ F_i &= \Phi \frac{\partial \phi}{\partial f_i} + \Phi_1 \frac{\partial \phi_1}{\partial f_i} + \ldots\ldots + \Phi_m \frac{\partial \phi_m}{\partial f_i} \quad (i = 1,\ 2,\ \ldots,\ m) \end{aligned}\right\},$$

$$\left.\begin{aligned} 0 &= \Phi \frac{\partial \phi}{\partial F_r} + \Phi_1 \frac{\partial \phi_1}{\partial F_r} + \ldots\ldots + \Phi_m \frac{\partial \phi_m}{\partial F_r} \quad (r = 1,\ 2,\ \ldots,\ m) \\ 0 &= \Phi \frac{\partial \phi}{\partial x_s} + \Phi_1 \frac{\partial \phi_1}{\partial x_s} + \ldots\ldots + \Phi_m \frac{\partial \phi_m}{\partial x_s} \quad (s = 2m+2,\ \ldots,\ p) \end{aligned}\right\}.$$

* If, for instance, there were two relations, then F and f could be expressed in terms of $f_1, \ldots, f_m, F_1, \ldots, F_m$; and thus the reduced form could be changed so as to contain only $2m$ variables and therefore be reducible to only m terms, contrary to hypothesis.

Eliminating from the last set of $m + (p - 2m - 1)$ equations the ratios $\Phi : \Phi_1 : \ldots\ldots\ldots : \Phi_m$, we have a number of Jacobian determinants equal to zero, from which the inference is that a functional relation

$$\Pi(f, f_1, \ldots, f_m, \phi, \phi_1, \ldots, \phi_m) = 0$$

subsists; and Π may be a quite arbitrary function. Then, as in § 113, it follows that

$$\Phi = \lambda \frac{\partial \Pi}{\partial \phi}, \ \Phi_1 = \lambda \frac{\partial \Pi}{\partial \phi_1}, \ \ldots\ldots\ldots, \ \Phi_m = \lambda \frac{\partial \Pi}{\partial \phi_m};$$

$$F = -\lambda \frac{\partial \Pi}{\partial f}, \ F_1 = -\lambda \frac{\partial \Pi}{\partial f_1}, \ \ldots\ldots\ldots, \ F_m = -\lambda \frac{\partial \Pi}{\partial f_m}.$$

When these are solved so as to give $\phi, \phi_1, \ldots, \phi_m$ the elements of the generalised solution, it appears that each of these elements is a function of the quantities f and F; and similarly for the coefficients Φ. Since Π is arbitrary, each such function is arbitrary; but all the arbitrary functions have their form determined by that of Π and so they are not independent of one another. The form of any one of them may be arbitrarily assumed—thus inversely determining the form of Π—and the forms of all the others are then determinate.

This is the generalisation of the assumed form; being the general form, it necessarily includes the simplest as a special case. Since, as has just been explained, one of the functions may be arbitrarily assigned and the rest will then be determinate, Clebsch takes as the foundation of the simplest reduced form the special assumption that the coefficient of one of the differential elements shall be unity. Thus the *normal reduced form* is

$$df + F_1 df_1 + F_2 df_2 + \ldots\ldots + F_m df_m,$$

where there is now no relation among the quantities f and F. When such a particular reduced form has been obtained, the natural generalisation is to another normal reduced form

$$d\phi + \Phi_1 d\phi_1 + \Phi_2 d\phi_2 + \ldots\ldots + \Phi_m d\phi_m,$$

which shall be the most general possible.

The preceding analysis will apply to this case, so that we have, in the first place, an arbitrary functional relation

$$\Pi(f, f_1, \ldots, f_m, \phi, \phi_1, \ldots, \phi_m) = 0$$

among the elements of the two solutions. But since F and Φ are each unity, we have

$$\lambda \frac{\partial \Pi}{\partial \phi} = 1 = -\lambda \frac{\partial \Pi}{\partial f},$$

and therefore

$$\frac{\partial \Pi}{\partial \phi} + \frac{\partial \Pi}{\partial f} = 0,$$

so that f and ϕ can enter into Π only in the combination $\phi - f$. Taking

$$\phi - f = \theta,$$

the arbitrary functional relation among the elements of the two solutions is

$$\left.\begin{array}{l} \Pi(f_1, \ldots, f_m, \theta, \phi_1, \ldots, \phi_m) = 0; \\ \text{and, if } \lambda \text{ be the reciprocal of } \dfrac{\partial \Pi}{\partial \theta}\text{, we have} \\ \qquad \Phi_r = \quad \lambda \dfrac{\partial \Pi}{\partial \phi_r} \; (r = 1, 2, \ldots, m) \\ \qquad F_i = -\lambda \dfrac{\partial \Pi}{\partial f_i} \; (i = 1, 2, \ldots, m) \end{array}\right\}.$$

These equations contain the *normal generalisation of any particular normal reduced form.*

128. When these equations are solved, they give each of the quantities $\theta, \phi_1, \ldots, \phi_m$ as a function of $f_1, \ldots, f_m, F_1, \ldots, F_m$; the forms of these functions are dependent on the form of Π, and so each function is arbitrary but is not independent of the others. Hence, if we regard only one of the elements of the generalised solution with the view of taking it as an integral, it follows that *the most general first integral of a differential equation with a reduced form*

$$df + F_1 df_1 + F_2 df_2 + \ldots\ldots + F_m df_m$$

is given by

$$\phi_m = \phi_m(f_1, \ldots, f_m, F_1, \ldots, F_m),$$

where ϕ_m is an arbitrary function.

The general march of the derivation of successive integrals is similar to that of the earlier case (§ 115). By means of the integral $\phi_m = a_m$, the differential equation can be transformed into

an equation containing only $p-1$ variables and having as a normal reduced form

$$d\phi + \Phi_1 d\phi_1 + \ldots\ldots + \Phi_{m-1} d\phi_{m-1};$$

as the new coefficients of the differential equation are affected by the form of ϕ_m adopted (or, on the other hand, as the remaining functions $\phi, \phi_1, \ldots, \phi_{m-1}$ are affected by the form of ϕ_m), it follows that the remaining integrals will be similarly affected. Hence, as in the earlier case, each integral affects the form of all those determined subsequently to itself.

The new equation, with the associated new reduced form, is the equation next simpler in treatment than the one already discussed; we shall have a second integral of the original equation in the form

$$\psi_{m-1} = \psi_{m-1}(\phi_1, \ldots, \phi_{m-1}, \Phi_1, \ldots \Phi_{m-1}),$$

where ψ_{m-1} is an arbitrary function. And then by means of the integral $\psi_{m-1} = a_{m-1}$, the differential equation can be transformed into one involving only $p-2$ variables and having

$$d\psi + \Psi_1 d\psi_1 + \ldots\ldots + \Psi_{m-2} d\psi_{m-2}$$

as a normal reduced form. And so on, until we obtain a differential equation in $p-m$ variables having

$$d\chi$$

as a normal reduced form, i.e. until we obtain an equation which is exact: then

$$\phi_m = a_m, \quad \psi_{m-1} = a_{m-1}, \quad \ldots\ldots, \quad \chi = a_0$$

constitute an integral system equivalent to the differential equation.

129. Having thus generalised any special solution, we have now to investigate the equations which determine the integrals. For this purpose Clebsch gives two processes, each devoted to the determination of a first integral of the differential equation. Since by means of this first integral the differential equation is transformed into another which is simpler and is similarly treated, it follows that each of these processes is limited in the same way as the first method which applies to the class of unconditioned equations in an even number of variables; neither of the processes possesses the general character of the second method which applies to that class of equations.

130. The first process is as follows. Since

$$\sum_{i=1}^{p} X_i dx_i = df + \sum_{r=1}^{m} F_r df_r,$$

we have

$$X_i = \frac{\partial f}{\partial x_i} + \sum_{r=1}^{m} F_r \frac{\partial f_r}{\partial x_i};$$

and therefore

$$a_{ij} = \sum_{r=1}^{m} \left(\frac{\partial F_r}{\partial x_j} \frac{\partial f_r}{\partial x_i} - \frac{\partial F_r}{\partial x_i} \frac{\partial f_r}{\partial x_j} \right),$$

which is the same in form as before (§ 118) and thus suggests a similar method.

Let $\lambda, \mu, \ldots, \rho$ be any $2m$ integers of the series $1, 2, \ldots, p$; let $p = 2m + q$, and let s be any one of the integers $1, 2, \ldots, q$. Then we determine q sets of quantities $\overset{(s)}{y}_1, \overset{(s)}{y}_2, \ldots, \overset{(s)}{y}_{2m}$ by the equations

$$\sum_{i=1}^{2m} a_{\theta,i} \overset{(s)}{y}_i = -a_{\theta,2m+s} \qquad (\theta = \lambda, \mu, \ldots, \rho).$$

Proceeding as in § 118, we obtain the equations—similar to (10) of that article—

$$\left.\begin{aligned} \overset{(s)}{y}_1 \frac{\partial f_r}{\partial x_1} + \overset{(s)}{y}_2 \frac{\partial f_r}{\partial x_2} + \ldots\ldots + \overset{(s)}{y}_{2m} \frac{\partial f_r}{\partial x_{2m}} + \frac{\partial f_r}{\partial x_{2m+s}} = 0 \\ \overset{(s)}{y}_1 \frac{\partial F_r}{\partial x_1} + \overset{(s)}{y}_2 \frac{\partial F_r}{\partial x_2} + \ldots\ldots + \overset{(s)}{y}_{2m} \frac{\partial F_r}{\partial x_{2m}} + \frac{\partial F_r}{\partial x_{2m+s}} = 0 \end{aligned}\right\},$$

holding for $r = 1, 2, \ldots, m$, and for $s = 1, 2, \ldots, q$.

Now it has been seen that the first integral of the differential equation is a function of $f_1, \ldots, f_m, F_1, \ldots, F_m$; hence it satisfies the equation

$$A_s \phi = \overset{(s)}{y}_1 \frac{\partial \phi}{\partial x_1} + \overset{(s)}{y}_2 \frac{\partial \phi}{\partial x_2} + \ldots\ldots + \overset{(s)}{y}_{2m} \frac{\partial \phi}{\partial x_{2m}} + \frac{\partial \phi}{\partial x_{2m+s}} = 0,$$

since each of its arguments satisfies the equation. This equation subsists for $s = 1, 2, \ldots, q$; and therefore we have the result:—

The first integral ϕ of the differential equation $\Omega = 0$ is determined as the most general simultaneous solution of the system of linear partial differential equations

$$A_1 \phi = 0, \quad A_2 \phi = 0, \quad \ldots\ldots, \quad A_q(\phi) = 0.$$

Each of the quantities y is the quotient of a Pfaffian of order $2m$ by the Pfaffian $[\lambda, \mu, \ldots, \rho]$; and as, by the initial hypothesis, not all the Pfaffians of order $2m$ vanish, we should naturally choose $\lambda, \mu, \ldots, \rho$ so as to give a non-vanishing Pfaffian.

It is easy to verify that the system of q equations satisfies all the Jacobian conditions for the possession of common solutions, and therefore it forms a complete system.

Ex. 1. The simplest case is that in which the number of characteristic equations is least, viz., $q=1$; and the single equation then serving to determine the first integral is

$$\sum_{s=1}^{2m+1} \frac{\partial \phi}{\partial x_s}[s+1, s+2, \ldots\ldots, 2m+1, 1, \ldots\ldots, s-1]=0.$$

As a very easy illustration, consider the equation

$$\Omega = x_2 dx_1 + x_3 dx_2 + x_1 dx_3 = 0.$$

The characteristic equation is

$$\frac{\partial \phi}{\partial x_1} + \frac{\partial \phi}{\partial x_2} + \frac{\partial \phi}{\partial x_3} = 0,$$

and therefore we may take as a special solution

$$f_1 = x_1 - x_2 = a.$$

Using this integral to remove x_1 and dx_1, we have

$$\begin{aligned}\Omega' &= x_2 dx_2 + x_3 dx_2 + (a + x_2)\, dx_3 \\ &= d\,(\tfrac{1}{2}x_2^2 + x_2 x_3 + a x_3),\end{aligned}$$

so that

$$f = \tfrac{1}{2}x_2^2 + x_2 x_3 + a x_3,$$

and therefore a *special* system of integrals is

$$f = \tfrac{1}{2}x_2^2 + x_1 x_3 = c, \quad f_1 = x_1 - x_2 = b.$$

Moreover, since

$$\Omega = df + F_1 df_1,$$

we find

$$F_1 = x_2 - x_3.$$

To *generalise the integral system* we take

$$\begin{aligned}\phi_1 &= \text{function } (f_1, F_1) \\ &= \text{function } (x_1 - x_2, x_2 - x_3) = \text{constant};\end{aligned}$$

and this may be taken

$$x_1 - x_2 = \psi\,(x_2 - x_3, a),$$

where ψ is an arbitrary function. Using this to modify the equation as before, we have

$$\begin{aligned}\Omega' &= x_2 dx_2 + x_3 \psi' dx_2 - x_3 \psi' dx_3 + x_3 dx_2 + x_2 dx_3 + \psi dx_3 \\ &= d\,(\tfrac{1}{2}x_2^2 + x_2 x_3 + x_3 \psi) + (x_2 - x_3)\, d\psi,\end{aligned}$$

so that, writing $x_2 - x_3 = z$, we have

$$f = \tfrac{1}{2}x_2^2 + x_1x_3 + \int z\psi'(z, a)\,dz,$$

where, after integration, z is to be replaced by its value $x_2 - x_3$. Thus the two integrals of the generalised system are

$$\left.\begin{aligned} \phi = \tfrac{1}{2}x_2^2 + x_1x_3 + \int z\psi'(z, a)\,dz = c \\ x_1 - x_2 = \psi(x_2 - x_3, a) \end{aligned}\right\}.$$

Ex. 2. Another easy illustration is afforded by the similar equation

$$x_2dx_1 + x_3dx_2 + x_4dx_3 + x_5dx_4 + x_1dx_5 = 0.$$

Ex. 3. It is an immediate corollary from Clebsch's first process that, if z and dz be eliminated from the non-integrable expression

$$\Omega = Xdx + Ydy + Zdz$$

by means of any integral, say $\phi(x, y, z, a) = 0$, of the characteristic equation

$$[23]\frac{\partial\phi}{\partial x_1} + [31]\frac{\partial\phi}{\partial x_2} + [12]\frac{\partial\phi}{\partial x_3} = 0,$$

then the new form Ω' ($=\Omega$ transformed) is a perfect differential. This result may be compared with Bertrand's method for an exact equation (§ 16), and with the theorem of Jacobi in the example in § 68.

Ex. 4. The equation

$$(x_2^2 - x_3x_1 + a)\,dx_1 + (x_3^2 - x_1x_2 + b)\,dx_2 + (x_1^2 - x_2x_3 + c)\,dx_3 = 0$$

will furnish in the following solution some variation in the merely analytical procedure.

We have $[12] = 3x_2$, $[23] = 3x_3$, $[31] = 3x_1$; so that, on the supposition that a, b, c do not all vanish, the equation is not exact. The characteristic equation for the first integral by Clebsch's first process is thus

$$x_3\frac{\partial\phi}{\partial x_1} + x_1\frac{\partial\phi}{\partial x_2} + x_2\frac{\partial\phi}{\partial x_3} = 0.$$

Equations subsidiary to its solution are

$$\frac{dx_1}{x_3} = \frac{dx_2}{x_1} = \frac{dx_3}{x_2},$$

two independent integrals of which are easily obtained in the forms

$$f = \frac{(x_1 + x_2 + x_3)^{\omega}}{x_1 + \omega x_2 + \omega^2 x_3}, \qquad \psi = \frac{(x_1 + x_2 + x_3)^{\omega^2}}{x_1 + \omega^2 x_2 + \omega x_3},$$

each equated to a constant, ω being a cube root of unity. Also

$$u = \frac{1}{f\psi} = x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3$$

is another integral; and, though neither f nor ψ can easily be used, yet u may be treated as a special first integral by means of which the original equation may be modified.

Instead of thus following the general rule, we proceed otherwise. It is not difficult to prove that

$$\frac{df}{f}+\omega\frac{d\psi}{\psi}=\frac{-3\omega^2}{u}\{(x_2^2-x_3x_1)\,dx_1+(x_3^2-x_1x_2)\,dx_2+(x_1^2-x_2x_3)\,dx_3\}.$$

Hence, taking

$$\vartheta=ax_1+bx_2+cx_3,$$

we have

$$\Omega=d\vartheta-\frac{\omega}{3f\psi}\,d\{\log(f\psi^{\omega})\};$$

and the special and general integrals can now be obtained by the usual process.

Ex. 5. Integrate the equation

$$x_2^2dx_1+x_3^2dx_2+x_1^2dx_3=0.$$

Ex. 6. It is easy to infer the result, slightly more general in form than that stated in Ex. 3, that, if α and β be two independent integrals of the system

$$\frac{dx_3}{[12]}=\frac{dx_1}{[23]}=\frac{dx_2}{[31]},$$

then the expression $X_1dx_1+X_2dx_2+X_3dx_3$ is equal to

$$d\gamma+Md\alpha+Nd\beta,$$

where M and N are functions of α and β alone, and γ is not determined by α and β*. If the condition of integrability be satisfied, then γ is a constant; and again we have Bertrand's theorem.

131. The second process given by Clebsch is as follows. The principle depends upon the fact that, when a normal reduced form

$$df+\sum_{i=1}^{m}F_i df_i$$

is transformed by means of a relation

$$f_m=\text{function}\,(f, f_1, \ldots, f_{m-1}, F_1, \ldots, F_m),$$

which it must be noticed involves f, it is changed into a form containing $2m$ independent quantities $f, f_1, \ldots, f_{m-1}, F_1, \ldots\ldots, F_m$ bound together by no relations. The new differential expression has thus a reduced form of the type

$$\sum_{i=1}^{m}F_i'\,df_i',$$

where the quantities f' and F' are connected by no relations; it therefore belongs to the class of equations treated already, (§§ 117, 118). Hence we have to find some function ϕ of the quantities $f, f_1, \ldots, f_m, F_1, \ldots, F_m$, which shall be an integral of the

* Darboux, "Sur le problème de Pfaff," *Bull. des Sci. Math.*, 2me Sér., t. vi., p. 24.

equation; and then, by means of the equation $\phi = a$, we remove one of the variables and obtain a new equation which, as has been seen, is amenable to the earlier methods.

For this purpose we again use the first of the propositions proved in § 121, in the following form. Consider $2m+2$ variables $x_1, \ldots, x_{2m+2}$ and $2m+2$ functions $f_0, f_1, \ldots, f_m, F_0, F_1, \ldots, F_m$, such that

$$\sum_{i=1}^{2m+2} X_i\, dx_i = \sum_{i=0}^{m} F_i\, df_i;$$

then the Jacobian

$$\frac{\partial(f_0, f_1, \ldots, f_m, F_0, F_1, \ldots, F_m)}{\partial(x_1, x_2, \ldots\ldots\ldots\ldots, x_{2m})}$$

has for its value the Pfaffian $P = [1, 2, \ldots, 2m+2]$. By the first of the propositions in § 121 we have

$$\sum_{h=1}^{2m+2} \sum_{k=1}^{2m+2} \frac{R_{h,k}}{P} X_k \frac{\partial\phi}{\partial x_h} = -\sum_{s=0}^{m} F_s \frac{\partial\phi}{\partial F_s},$$

where ϕ is any function of the $2m+2$ variables. But, by the ordinary formulæ for Jacobians, we have

$$\begin{aligned} P\frac{\partial\phi}{\partial F_s} &= \frac{\partial(f_0, f_1, \ldots, f_m, F_0, F_1, \ldots, F_m)}{\partial(x_1, x_2, \ldots\ldots\ldots\ldots x, {}_{2m+2})}\frac{\partial\phi}{\partial F_s} \\ &= \frac{\partial(f_0, f_1, \ldots, f_m, F_0, F_1, \ldots, F_{s-1}, \phi, F_{s+1}, \ldots, F_m)}{\partial(x_1, x_2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, x_{2m+2})}; \end{aligned}$$

and therefore

$$\begin{aligned} &\sum_{h=1}^{2m+2} \sum_{k=1}^{2m+2} R_{h,k} X_k \frac{\partial\phi}{\partial x_h} \\ &\qquad = -\sum_{s=0}^{m} F_s \frac{\partial(f_0, f_1, \ldots, f_m, F_0, F_1, \ldots, F_{s-1}, \phi, F_{s+1}, \ldots, F_m)}{\partial(x_1, x_2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, x_{2m+2})}. \end{aligned}$$

To adapt this, which is a general result, to the case of the normal form under consideration, we notice that F_0 is unity and that therefore every term on the right-hand side will vanish, except that in which ϕ replaces F_0: hence, whatever the function ϕ may be, as a function of the variables, we have

$$\sum_{h=1}^{2m+2} \sum_{k=1}^{2m+2} R_{h,k} X_k \frac{\partial\phi}{\partial x_h} = -\frac{\partial(f, f_1, \ldots, f_m, \phi, F_1, \ldots, F_m)}{\partial(x_1, x_2, \ldots\ldots\ldots\ldots, x_{2m+2})}.$$

And it is evident that a similar formula will hold, *mutatis mutandis*, for every selection of $2m+2$ integers from the series $1, 2, \ldots, p$.

Now when ϕ, a function of the $2m+2$ variables, can be expressed in terms of $f, f_1, \ldots, f_m, F_1, \ldots, F_m$ alone, the Jacobian of the $2m+2$ functions vanishes; and when ϕ, considered as a function of all the p variables, is expressible in terms of $f, f_1, \ldots, f_m, F_1, \ldots, F_m$ alone, every Jacobian of the $2m+2$ functions, taken with reference to every set of $2m+2$ variables chosen from the p variables, vanishes. Hence the desired function ϕ satisfies all the possible differential equations of the form

$$\sum_{h=1}^{2m+2} \sum_{k=1}^{2m+2} R_{h,k} X_k \frac{\partial \phi}{\partial x_h} = 0.$$

The desired functional dependence of ϕ on the quantities f and F is sufficiently and completely established by the vanishing of the Jacobians of the $2m+2$ functions taken with reference to $x_1, x_2, \ldots, x_{2m+1}$ and each of the other variables in turn. If then $R^{(s)}_{h,k}$ denote the derivative of $[1, 2, \ldots, 2m+1, 2m+s]$ with regard to h, k; and if

$$z^{(s)}_h = \sum_{i=1}^{2m+1} X_i R^{(s)}_{h,i} + X_{2m+s} R^{(s)}_{h,2m+s},$$

then *the partial differential equations necessary and sufficient to determine the function* ϕ *are*

$$A^{(s)} \phi = \sum_{h=1}^{2m+1} z^{(s)}_h \frac{\partial \phi}{\partial x_h} + z^{(s)}_{2m+s} \frac{\partial \phi}{\partial x_{2m+s}} = 0,$$

for $s = 1, 2, \ldots, p-2m-1$.

In the particular case of an unconditioned equation in an odd number of variables we have $p = 2m+1$, so that the preceding analysis does not apply. But then, as there are only $2m+1$ variables and there are $2m+1$ quantities f and F, it follows that any function ϕ whatever can be expressed in terms of f and F; hence a first integral would be obtained by equating to a constant any arbitrary function of the variables and using it as indicated in § 118. This is the theory previously adopted.

It is easy to verify for the general case that the system of $p-2m-1$ equations satisfies all the Jacobian conditions for the possession of common solutions, and therefore it forms a complete system.

Ex. Integrate the equation

$$(2y+2z-u)\,dx + (z+v)\,dy + (y+v)\,dz + (x-v)\,du + (u-y-z)\,dv = 0,$$

which has a normal reduced form $d\theta + F df$.

CHAPTER IX.

Tangential Transformations.

For the history of tangential transformations see Lie, *Math. Ann.*, t. viii., p. 219 and Mansion "Théorie des équations aux dérivées partielles du premier ordre" p. 42. In addition to the memoir of Lie (which is a résumé of several memoirs published earlier in Christiania) and that of Mayer, both of which are quoted in this chapter, reference may be made to a memoir by Lie, *Arch. for Math. og Nat.*, t. ii. (1877), pp. 10—38 and to one by Engel, *Math. Ann.*, t. xxiii., pp. 1—44.

An exhaustive discussion of the theory of tangential transformations with its present developments is to be found in the second volume of Lie and Engel's "Theorie der Transformationsgruppen" (Leipzig, 1890).

In a note *Math. Ann.*, t. viii., p. 223 Lie suggested questions relative to tangential transformations and osculational transformations which had already engaged the attention of Bäcklund. The latter had published a memoir "Einiges über Curven- und Flächentransformationen," *Lunds Arsskrift*, t. x. (1875); and since that date he has published memoirs on the subject in the *Math. Ann.*, t. ix., pp. 297—320; *ib.*, t. xi., pp. 199—241 especially pp. 200—219; *ib.*, t. xix., pp. 387—422.

132. The general idea of the class of transformations about to be considered is derived from certain fundamental geometrical ideas relative to the transformations of space. Transformations between the points of different spaces will in general change aggregates of points into aggregates of points with similar characteristic features; for example, a surface in geometry of three dimensions is changed into a surface, two surfaces which touch one another are changed into two others which touch one another. But point-and-point transformations do not alone lead to such results; thus the tangential property indicated is possessed also by the dualistic transformations which analytically express reciprocation with regard to a quadric; and these are only two sets from an extensive class.

All transformations, characterised by the property of changing surfaces in one space which touch one another into surfaces in another space which touch one another, are called ***tangential transformations****; they are of course not limited to the analytical forms which correspond to geometrical space, but they apply to the most general configurations in an amplitude (Mannigfaltigkeit) of any number of dimensions.

Suppose then that there is an amplitude of $n+1$ dimensions and that any point in it is determined by the (non-homogeneous) coordinates $z, x_1, x_2, \ldots, x_n$; and in it let there be two configurations (algebraische Gebilde), each of n dimensions, which touch one another. At the common point the quantities $z, x_1, \ldots, x_n, p_1, \ldots, p_n$ are the same for both configurations; and the contact at that common point is ensured, if the differential relation

$$dz - \sum_{i=1}^{n} p_i dx_i = 0,$$

which subsists for all variations in one of the configurations at the point, subsist also for all variations in the other at the point with the same finite elements. Let there be a transformation which gives $z', x_1', \ldots, x_n'$ as the coordinates of a point in another amplitude or in another part of the former one; the two configurations, which are the two earlier configurations transformed, will touch one another at a common point, if the differential relation

$$dz' - \sum_{i=1}^{n} p_i' dx_i' = 0$$

subsist for the two configurations simultaneously. It thus follows that two touching configurations are transformed into other two touching configurations when the vanishing of $dz' - \sum_{i=1}^{n} p_i' dx_i'$ is consequent on the vanishing of $dz - \sum_{i=1}^{n} p_i dx_i$; hence *the transformation must be* such as to permit these quantities to vanish together and therefore *such as to require an identical relation*

$$dz' - \sum_{i=1}^{n} p_i' dx_i' = \rho \left(dz - \sum_{i=1}^{n} p_i dx_i \right),$$

* They are called by Lie and others *Berührungstransformationen*; by French writers *transformations de contact*.

where ρ is a non-vanishing integral quantity. Moreover, the quantities $z, x_1, \ldots, x_n, p_1, \ldots, p_n$ are the coordinates of an element of a configuration of the most extensive type; the quantities $z, x_1, \ldots, x_n$ determine its position and the quantities $p_1, \ldots, p_n$ its orientation in that position; hence the $2n+1$ quantities are independent of one another*.

Hence we are led to Lie's definition† of a tangential transformation:—

When $Z, X_1, \ldots, X_n, P_1, \ldots, P_n$ are $2n+1$ independent functions of the $2n+1$ independent quantities $z, x_1, \ldots, x_n, p_1, \ldots, p_n$ such that the variational relation

$$dZ - \sum_{i=1}^{n} P_i dX_i = \rho\left(dz - \sum_{i=1}^{n} p_i dx_i\right)$$

(where ρ does not vanish) is identically satisfied, then the transformation defined by the equations

$$z' = Z, \quad x' = X, \quad p' = P$$

is called a tangential transformation.

Two classes of tangential transformation, viz., point-and-point transformation and the (reciprocal polar) point-and-plane transformation, have already been referred to; it is of importance to determine all the classes. The analytical problem thus presented amounts to the determination of Z, X, P as independent functions of z, x, p in the most general form which admits of the characteristic variational relation being identically satisfied.

133. Since the quantities in the two differential expressions which occur in the variational relation are two aggregates of independent quantities, each of the expressions may be regarded as a normal reduced form of an unconditioned Pfaffian differential expression involving $2n+1$ variables; so that Clebsch's investigations (§§ 113, 127) relative to the derivation of the most

* Thus z, x, y in ordinary space determine a point on a surface; p and q determine the position of an element at the point; so that any element of any surface is defined by the five quantities z, x, y, p, q, and the whole of any surface containing a given element would be defined by a single relation between the five quantities, that is, by a partial differential equation. It is easy to infer the corresponding results for configurations of less extensive type, e.g., for tortuous curves in ordinary space.

† *Math. Ann.*, t. viii., p. 220.

general normal form from a given normal form may be applied to the present question. It is thus, in fact, that Lie discusses it*; he regards it as a special case of Pfaff's problem and applies Clebsch's results. But as some of the special inferences made by Lie in the theory of tangential transformations are applied by him (of course not unjustifiably) to the discussion of Pfaff's problem, it is (as also for other reasons) of some advantage to have an independent establishment of the principal results. This direct derivation of the fundamental formulæ of the theory has been effected by Mayer†.

134. We have then to determine $2n+1$ algebraically independent quantities Z, X, P as functions of $2n+1$ independent variables z, x, p such that the equation

$$dZ - \sum_{i=1}^{n} P_i dX_i = \rho\left(dz - \sum_{i=1}^{n} p_i dx_i\right) \qquad \text{.............(1)}$$

is identically satisfied, ρ being a non-vanishing quantity. Since the variables are independent, the equation (1) is equivalent to the set of $2n+1$ equations constituted by

$$\frac{\partial Z}{\partial z} - \sum_{i=1}^{n} P_i \frac{\partial X_i}{\partial z} = \rho \qquad \text{.....................(2)},$$

$$B_r = \frac{\partial Z}{\partial p_r} - \sum_{i=1}^{n} P_i \frac{\partial X_i}{\partial p_r} = 0 \qquad \text{.....................(3)},$$

$$\frac{\partial Z}{\partial x_r} - \sum_{i=1}^{n} P_i \frac{\partial X_i}{\partial x_r} = -\rho p_r \qquad \text{...............(4)}',$$

the equations (3) and (4)′ holding for $r = 1, 2, \ldots, n$. Taking as a new symbol $\dfrac{dU}{dx_r}$ to represent

$$\frac{\partial U}{\partial x_r} + p_r \frac{\partial U}{\partial z}$$

for any function U, the n equations (4)′ may by means of (2) be replaced by the n equations

$$A_r = \frac{dZ}{dx_r} - \sum_{i=1}^{n} P_i \frac{dX_i}{dx_r} = 0 \qquad \text{..................(4)},$$

holding for $r = 1, 2, \ldots, n$; and these equations (2), (3) and (4)

* *Math. Ann.*, t. viii., pp. 221 et seq.

† *Gött. Nach.* (1874), pp. 317—331; reproduced, *Math. Ann.*, t. viii., pp. 304—312.

are sufficient to ensure the satisfaction of the fundamental equation (1).

There are two stages in the investigation; the first is the establishment of necessary equations determining Z, X, P; the second is the selection, from these necessary equations, of such as are independent and sufficient for the determination.

For the first, we have for any function U

$$\frac{\partial}{\partial p_r}\frac{\partial U}{\partial p_s} - \frac{\partial}{\partial p_s}\frac{\partial U}{\partial p_r} = 0,$$

$$\left.\begin{aligned} \frac{\partial}{\partial p_r}\frac{dU}{dx_r} - \frac{d}{dx_r}\frac{\partial U}{\partial p_r} &= \frac{\partial U}{\partial z}, \\ \frac{\partial}{\partial p_r}\frac{dU}{dx_s} - \frac{d}{dx_s}\frac{\partial U}{\partial p_r} &= 0, \text{ if } r \gtrless s, \end{aligned}\right\}$$

$$\frac{d}{dx_r}\frac{dU}{dx_s} - \frac{d}{dx_s}\frac{dU}{dx_r} = 0.$$

Then

$$\frac{\partial}{\partial p_r}\sum_{i=1}^{n} P_i \frac{\partial X_i}{\partial p_s} = \frac{\partial^2 Z}{\partial p_r \partial p_s}$$

$$= \frac{\partial}{\partial p_s}\sum_{i=1}^{n} P_i \frac{\partial X_i}{\partial p_r},$$

by taking $U = Z$ in the first of the foregoing relations and using (3). When this is simplified, it gives

$$\sum_{i=1}^{n}\left(\frac{\partial P_i}{\partial p_r}\frac{\partial X_i}{\partial p_s} - \frac{\partial P_i}{\partial p_s}\frac{\partial X_i}{\partial p_r}\right) = 0.$$

Similarly using the other relations in connection with equations (3) and (4), we obtain the aggregate of equations

$$\left.\begin{aligned} \sum_{i=1}^{n}\left(\frac{\partial P_i}{\partial p_r}\frac{dX_i}{dx_r} - \frac{dP_i}{dx_r}\frac{\partial X_i}{\partial p_r}\right) &= \rho, \\ \sum_{i=1}^{n}\left(\frac{\partial P_i}{\partial p_r}\frac{dX_i}{dx_s} - \frac{dP_i}{dx_s}\frac{\partial X_i}{\partial p_r}\right) &= 0, \text{ if } r \gtrless s, \\ \sum_{i=1}^{n}\left(\frac{dP_i}{dx_r}\frac{dX_i}{dx_s} - \frac{dP_i}{dx_s}\frac{dX_i}{dx_r}\right) &= 0. \end{aligned}\right\} \quad \ldots\ldots(5).$$

Introducing now a set of $2n$ independent quantities $y_1, \ldots, y_n$, $z_1, \ldots, z_n$ and another set of $2n$ quantities $u_1, \ldots, u_n, v_1, \ldots v_n$, connected by the linear equations

$$\left.\begin{aligned} u_j &= \sum_{i=1}^{n} \left(\frac{dX_j}{dx_i} y_i + \frac{\partial X_j}{\partial p_i} z_i\right) \\ v_j &= \sum_{i=1}^{n} \left(\frac{dP_j}{dx_i} y_i + \frac{\partial P_j}{\partial p_i} z_i\right) \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots(6),$$

then we have uniquely, by the use of (5), the complementary equations

$$\left.\begin{aligned} \sum_{j=1}^{n} \left(u_j \frac{\partial P_j}{\partial p_i} - v_j \frac{\partial X_j}{\partial p_i}\right) &= \rho y_i \\ \sum_{j=1}^{n} \left(u_j \frac{dP_j}{dx_i} - v_j \frac{dX_j}{dx_i}\right) &= -\rho z_i \end{aligned}\right\} \ldots\ldots\ldots\ldots(7).$$

If, instead of using (5) to find y and z in terms of u and v, we solve (6) to obtain their values, it follows, since the quantities y and z are independent, that the quantities u and v are connected by no linear relations unless the determinant R of the coefficients on the right-hand sides of (6) vanishes. And from (7) it follows, with the same suppositions, that R (also the determinant of the coefficients on the left-hand sides) vanishes only with ρ, which has been supposed a non-vanishing quantity; so that u and v are a set of $2n$ independent quantities.

Further, taking the Jacobian Θ of Z, X, P with regard to z, x, p and substituting from (2), (3), (4)′ for $\frac{\partial Z}{\partial z}$, $\frac{\partial Z}{\partial p}$, $\frac{\partial Z}{\partial x}$, we easily find

$$\Theta = \rho R,$$

which thus does not vanish; and therefore Z, X, P are functionally independent of one another.

Substituting now the values of y_i and z_i as given by equations (7) in equations (6), we have

$$\rho u_j = \sum_{i=1}^{n} \left\{\frac{dX_j}{dx_i} \sum_{s=1}^{n} \left(u_s \frac{\partial P_s}{\partial p_i} - v_s \frac{\partial X_s}{\partial p_i}\right) - \frac{\partial X_j}{\partial p_i} \sum_{s=1}^{n} \left(u_s \frac{dP_s}{dx_i} - v_s \frac{dX_s}{dx_i}\right)\right\},$$

$$\rho v_j = \sum_{i=1}^{n} \left\{\frac{dP_j}{dx_i} \sum_{s=1}^{n} \left(u_s \frac{\partial P_s}{\partial p_i} - v_s \frac{\partial X_s}{\partial p_i}\right) - \frac{\partial P_j}{\partial p_i} \sum_{s=1}^{n} \left(u_s \frac{dP_s}{dx_i} - v_s \frac{dX_s}{dx_i}\right)\right\},$$

which must be mere identities since there are no linear relations

among the quantities u and v. Hence, using the symbol $[F\Phi]$ to denote

$$\sum_{i=1}^{n}\left(\frac{\partial F}{\partial p_i}\frac{d\Phi}{dx_i}-\frac{dF}{dx_i}\frac{\partial\Phi}{\partial p_i}\right),$$

we have

$$\left.\begin{aligned} \rho &= [P_j X_j], \text{ for } j=1, \dots, n \\ 0 &= [P_i X_j], \text{ if } i \gtrless j \\ 0 &= [X_i X_j], \quad \dots \\ 0 &= [P_i P_j], \quad \dots \end{aligned}\right\} \dots\dots\dots\dots\dots(8),$$

which are *necessary* conditions affecting the quantities X, P.

Secondly, if Θ be any function whatever of z, x, p, we have, on substituting the expressions for A_r and B_r, the relation

$$\sum_{r=1}^{n}\left(B_r\frac{d\Theta}{dx_r}-A_r\frac{\partial\Theta}{\partial p_r}\right)=[Z\Theta]+\sum_{j=1}^{n}P_j[\Theta X_j]\dots\dots(9);$$

and therefore by the relations (8)

$$\left.\begin{aligned} \sum_{r=1}^{n}\left(B_r\frac{dX_i}{dx_r}-A_r\frac{\partial X_i}{\partial p_r}\right)&=[ZX_i] \\ \sum_{r=1}^{n}\left(B_r\frac{dP_i}{dx_r}-A_r\frac{\partial P_i}{\partial p_r}\right)&=[ZP_i]+\rho P_i \end{aligned}\right\}\dots\dots\dots\dots(10).$$

In these equations (10), the determinant of the coefficients on the left-hand sides is not zero; and therefore, if we take

$$\left.\begin{aligned} [ZX_i]&=0 \\ \rho P_i+[ZP_i]&=0 \end{aligned}\right\}, \text{ for } i=1, 2, \dots, n,$$

the equations (10) lead to $A_r=0$, $B_r=0$, that is, they give equations (3) and (4) uniquely. Hence

$$[ZX_i]=0, \quad [ZP_i]=-\rho P_i,$$

are *sufficient* to ensure the existence of (3) and (4), which with (2) are equivalent to the fundamental relation.

The combined results now obtained lead to Lie's theorem:—

In order that the relation

$$dZ-\sum_{i=1}^{n}P_i dX_i=\rho\left(dz-\sum_{i=1}^{n}p_i dx_i\right)$$

may be identically satisfied, it is necessary and sufficient that the quantities Z, X, P satisfy the equations

$$\left.\begin{array}{c}[ZX_i]=0=[X_iX_j]=[P_iX_j]=[P_iP_j]\\ {[P_jX_j]=\rho, \quad [ZP_i]=-\rho P_i}\end{array}\right\},$$

provided Z, X, P are functionally independent of one another; and the value of ρ is

$$\frac{\partial Z}{\partial z}-\sum_{i=1}^{n} P_i\frac{\partial X_i}{\partial z},$$

a non-vanishing quantity. Also, conversely, a non-zero value of ρ is sufficient to ensure the functional independence of Z, X, P.

135. The aggregate of the conditions—each of them a differential equation—in Lie's theorem is large; and, though they are necessary and sufficient, no indication is given of any of them being superfluous because not independent. To modify the result and reduce the number of sufficient and necessary conditions as far as possible, Mayer proceeds as follows.

From (9) it follows, by taking Θ equal to Z and to X_i, that

$$\sum_{r=1}^{n}\left(B_r\frac{dZ}{dx_r}-A_r\frac{\partial Z}{\partial p_r}\right)=\sum_{j=1}^{n}P_j[ZX_j],$$

$$\sum_{r=1}^{n}\left(B_r\frac{dX_i}{dx_r}-A_r\frac{\partial X_i}{\partial p_r}\right)=[ZX_i]+\sum_{j=1}^{n}P_j[X_iX_j];$$

and therefore, if there be $n+1$ algebraically independent functions Z, $X_1, \ldots, X_n$ which satisfy the equations

$$[ZX_i]=0, \quad [X_iX_j]=0,$$

there are apparently $n+1$ equations, n of which are of the form

$$\sum_{r=1}^{n}\left(B_r\frac{dX_i}{dx_r}-A_r\frac{\partial X_i}{\partial p_r}\right)=0 \text{ for } i=1, \ldots, n,$$

the remaining equation being

$$\sum_{r=1}^{n}\left(B_r\frac{dZ}{dx_r}-A_r\frac{\partial Z}{\partial p_r}\right)=0.$$

The last one however may be disregarded when the earlier n are retained: for, multiplying those n equations by $P_1, \ldots, P_n$ and adding, we have

$$\sum_{r=1}^{n}\left\{B_r\left(\frac{dZ}{dx_r}-A_r\right)-A_r\left(\frac{\partial Z}{\partial p_r}-B_r\right)\right\}=0,$$

which at once gives the last equation. We thus have n equations linear and homogeneous in the $2n$ quantities A and B; and, if n of the quantities vanish, the n equations then involve the other n linearly and homogeneously. Now the determinants of their coefficients do not all vanish, for otherwise there would be a functional relation among the quantities Z and X; hence the n equations can only be satisfied by having the n variables occurring in them zero. Hence it follows that, *when the* $n+1$ *quantities* Z *and* X *have been determined as functionally independent solutions of the equations*

$$[ZX_i]=0,\ [X_iX_j]=0,$$

then n *of the equations* (3) *and* (4) *can be derived from the other* n, *so that the set is equivalent to only* n *independent equations. These equations* serve to determine the* n *quantities* $P_1, \ldots, P_n$; *and the value of* ρ *is then determined by equation* (2).

This is Mayer's form of Lie's theorem.

136. The quantities occurring in the general tangential transformation are $2n+2$ in number, viz., ρ, Z, $X_1, \ldots, X_n$, $P_1, \ldots, P_n$; and there are only $2n+1$ equations to determine them, viz., (2), (3), (4). Hence some one of the quantities will be left undetermined; and, unless an external condition be assigned, an arbitrary element will occur in the solution.

This expectation is verified in the character of the result just obtained; for the equations which determine the quantities Z and X are

$$[ZX_i]=0, \quad [X_iX_j]=0,$$

and the first of the quantities thus determined may be taken at pleasure, its form affecting that of the others subsequently determined.

Ex. An important application of this result has been made† relative to the solution of partial differential equations. (See also chap. vii.) So far as

* It may happen that some set of n equations chosen from the $2n$ may have, owing to the values of Z and X, the determinant of the coefficients P zero: but this will not occur for all such sets, on account of the functional independence of Z and X.

† Darboux, *Bull. des Sc. Math.*, 2me Sér. t. vi. (1882), p. 67 for the present form; but essentially the application is Lie's, *Math. Ann.*, t. viii. (1875), p. 242, also p. 311.

the preceding general investigation is concerned, the quantities $p_1, \ldots\ldots, p_n$ are algebraically independent of one another and of $x_1, \ldots\ldots, x_n$; and the characteristic relation

$$dZ - \sum_{i=1}^{n} P_i dX_i = \rho (dz - \sum_{i=1}^{n} p_i dx_i)$$

is identically satisfied, some one of the quantities Z and X being undetermined by the conditions.

Let then $p_1, \ldots\ldots, p_n$ be the derivatives of z, a supposition which does not violate the general condition; and let a given differential equation of the first order be

$$Z = 0.$$

Then we take Z as the arbitrary element just indicated, and we determine the quantities X by the equations

$$[ZX_1] = 0, \ldots\ldots\ldots\ldots\ldots, [ZX_n] = 0,$$
$$[X_i X_j] = 0,$$

which occur in the ordinary Jacobian method of solution. Since

$$dz = \sum_{i=1}^{n} p_i dx_i$$

and

$$dZ = 0,$$

the characteristic relation leads to the equation

$$\sum_{i=1}^{n} P_i dX_i = 0.$$

Then the equations

$$X_1 = a_1,\ X_2 = a_2, \ldots\ldots, X_n = a_n,$$

combined with $Z=0$, give on the elimination of $p_1, \ldots\ldots, p_n$ a complete integral of $Z=0$. And, as remarked by Jordan *, the equations

$$P_1 = 0,\ P_2 = 0, \ldots\ldots, P_n = 0$$

give a singular integral when combined with $Z=0$. The most comprehensive general integral is given by

$$\left.\begin{array}{c} f(X_1, \ldots\ldots, X_n) = 0,\ Z = 0 \\ \dfrac{1}{P_1}\dfrac{\partial f}{\partial X_1} = \ldots\ldots\ldots\ldots = \dfrac{1}{P_n}\dfrac{\partial f}{\partial X_n} \end{array}\right\},$$

where f is an arbitrary function.

137. The preceding theory gives the most general class of tangential transformations. There are several special classes, which are of importance: in particular, there are all those constituted by the transformations which give values of $X_1, \ldots, X_n, P_1, \ldots, P_n$ independent of z. These may be called cylindrical tangential transformations.

* *Cours d'Analyse*, t. iii. (1887), p. 340.

By Lie's theorem we have

$$\rho = [P_i X_i];$$

since P and X are independent of z, it follows that ρ is independent of z; and from equation (2), which now is

$$\frac{\partial Z}{\partial z} = \rho,$$

we have

$$Z = \rho z + \Pi,$$

where Π, independent of z, may be some function of $x_1, \ldots, x_n$, $p_1, \ldots, p_n$. From (4)′ we have

$$\begin{aligned} -\rho p_r + \sum_{i=1}^{n} P_i \frac{\partial X_i}{\partial x_r} &= \frac{\partial Z}{\partial x_r} \\ &= z \frac{\partial \rho}{\partial x_r} + \frac{\partial \Pi}{\partial x_r} \end{aligned}$$

for $r = 1, \ldots, n$; and since the only term in this equation which involves z is that occurring on the right-hand side, we have

$$\frac{\partial \rho}{\partial x_r} = 0$$

for $r = 1, \ldots, n$; or ρ is independent of the variables x. Similarly from (3), which is

$$\begin{aligned} \sum_{i=1}^{n} P_i \frac{\partial X_i}{\partial p_r} &= \frac{\partial Z}{\partial p_r} \\ &= z \frac{\partial \rho}{\partial p_r} + \frac{\partial \Pi}{\partial p_r}, \end{aligned}$$

it is inferred that ρ is independent of the variables p. Hence ρ is a constant: let its value be denoted by A; then the form of Z is

$$Z = Az + \Pi,$$

where Π is a function of x and p only*. Substituting this value of Z in the characteristic variational relation, we have

$$Adz + d\Pi - \sum_{i=1}^{n} P_i dX_i = A\left(dz - \sum_{i=1}^{n} p_i dx_i\right);$$

* The constant A can evidently be absorbed into the variables Z and X, if desirable; the transformation would then be

$$z' - z = \Pi,$$

a form which suggested the proposed name.

and therefore

$$d\Pi - \sum_{i=1}^{n} P_i dX_i = -A \sum_{i=1}^{n} p_i dx_i \qquad \text{..............(11)},$$

in which z no longer occurs.

The modified forms of the equations which determine Π, P, X are easily obtained. We have

$$\frac{d}{dx} X = \frac{\partial X}{\partial x}, \quad \frac{d}{dx} P = \frac{\partial P}{\partial x}, \quad \frac{\partial}{\partial p} Z = \frac{\partial \Pi}{\partial p},$$

$$\frac{d}{dx_r} Z = \frac{\partial \Pi}{\partial x_r} + A p_r;$$

and therefore, when we take

$$(F\Phi) = \sum_{r=1}^{n} \left(\frac{\partial F}{\partial p_r} \frac{\partial \Phi}{\partial x_r} - \frac{\partial F}{\partial x_r} \frac{\partial \Phi}{\partial p_r} \right),$$

we have

$$[X_i X_j] = (X_i X_j), \; [X_i P_j] = (X_i P_j), \; [P_i P_j] = (P_i P_j),$$

$$[ZP_i] = (\Pi P_i) - A \sum_{r=1}^{n} p_r \frac{\partial P_i}{\partial p_r},$$

$$[ZX_i] = (\Pi X_i) - A \sum_{r=1}^{n} p_r \frac{\partial X_i}{\partial p_r};$$

and so we have the theorem:—

In order that the relation

$$d\Pi - \sum_{i=1}^{n} P_i dX_i = -A \sum_{i=1}^{n} p_i dx_i,$$

where A is a constant and Π is a function of $x_1, \ldots, x_n, p_1, \ldots, p_n$, may be identically satisfied, it is necessary and sufficient that the quantities Π, P, X satisfy the equations

$$\left.\begin{aligned} (X_i X_j) = 0 = (P_i X_j) &= (P_i P_j) \\ (P_i X_i) &= A \\ (\Pi P_i) &= A \left(-P_i + \sum_{r=1}^{n} p_r \frac{\partial P_i}{\partial p_r} \right) \\ (\Pi X_i) &= A \sum_{r=1}^{n} p_r \frac{\partial X_i}{\partial p_r} \end{aligned}\right\};$$

and the quantities X and P are functionally independent of one another.

Or, if we take the second form (§ 135) of Lie's general theorem, we have for the present case:—

When the $n+1$ *quantities* Π *and* X *have been determined as functionally independent solutions of the equations*

$$(X_i X_j) = 0,$$

$$(\Pi X_i) = A \sum_{r=1}^{n} p_r \frac{\partial X_i}{\partial p_r},$$

then the quantities P *can be obtained from any* n *independent equations of the set*

$$\frac{\partial \Pi}{\partial x_i} - \sum_{r=1}^{n} P_r \frac{\partial X_r}{\partial x_i} = -Ap_i, \quad \frac{\partial \Pi}{\partial p_i} - \sum_{r=1}^{n} P_r \frac{\partial X_r}{\partial p_i} = 0.$$

138. *Ex.* 1. Some simple cases of cylindrical tangential transformations are given by Lagrange*, viz:—

(i) $Z = z - px - qy; \quad \left.\begin{array}{l} X = p, \; Y = q \\ P = -x, \; Q = -y \end{array}\right\},$

which is often called the Legendrian transformation:

(ii) $Z = z - qy \quad ; \quad \left.\begin{array}{l} X = x, \; Y = q \\ P = p, \; Q = -y \end{array}\right\};$

(iii) $Z = z - px \quad ; \quad \left.\begin{array}{l} X = p, \; Y = y \\ P = -x, \; Q = q \end{array}\right\}.$

They are used by Lagrange to solve particular classes of equations by transforming them into other known equations.

Ex. 2. Two important inferences, used by Lie in his theory of Pfaff's problem, may be made as follows.

First, in equation (11) the quantities P and X are $2n$ independent functions of the variables $x_1, \ldots\ldots, x_n, p_1, \ldots\ldots, p_n$; and Π is another function of the same variables, such that there exists a single relation among the quantities Π, P, X. Moreover, the constant $-A$ in that equation can be absorbed into the left-hand side; and hence we have the theorem:—

I. *When a single relation exists among* $2n+1$ *quantities* $y_0, y_1, \ldots\ldots, y_n, q_1, \ldots\ldots, q_n$, *then the expression*

$$dy_0 + \sum_{i=1}^{n} q_i dy_i$$

can be transformed to

$$\sum_{i=1}^{n} p_i dx_i.$$

(This result is a particular case of Pfaff's reduction of a differential expression containing an even number of variables: for, since y_0 is a function

* *Oeuvres complètes*, t. iv., p. 84.

of q and y, the differential expression will, after substitution is made for y_0, contain $2n$ differential elements and variables and the new differential expression is merely its reduced normal form.)

A simpler form of the result just obtained arises by taking $A=1$ in equation (11), when we have

$$d\Pi + \sum_{i=1}^{n} p_i dx_i = \sum_{i=1}^{n} P_i dX_i \qquad (a).$$

When we consider Π as a given function of x and p, the equations determining the quantities X are

$$\left.\begin{aligned}(\Pi X_i) &= \sum_{r=1}^{n} p_r \frac{\partial X_i}{\partial p_r}\\ (X_i X_j) &= 0\end{aligned}\right\} \qquad (b);$$

and the quantities P are determined as in the second form of the theorem of § 137.

Second, a special supposition with regard to equation (a) just obtained leads to another reduction. Suppose it possible to have

$$X_n = x_n;$$

let us find the form of Π and the associated limitations on the quantities P and X.

With the given value of X_n, we have

$$0 = (X_i X_n) = \frac{\partial X_i}{\partial p_n},$$

so that X_i is independent of p_n. Also, for $i = 1, 2, \ldots\ldots, n-1$, we have

$$0 = (P_i X_n) = \frac{\partial P_i}{\partial p_n},$$

so that $P_1, \ldots\ldots, P_{n-1}$ are independent of p_n; and we have

$$1 = (P_n X_n) = \frac{\partial P_n}{\partial p_n},$$

so that
$$P_n = p_n + \Theta,$$

where Θ is independent of p_n. Lastly, we have from the equation

$$(\Pi X_n) = \sum_{r=1}^{n} p_r \frac{\partial X_n}{\partial p_r}$$

the relation

$$\frac{\partial \Pi}{\partial p_n} = 0,$$

or Π is independent of p_n. The equation (a) now becomes

$$\begin{aligned} d\Pi + \sum_{i=1}^{n-1} p_i dx_i &= \sum_{i=1}^{n} P_i dX_i - p_n dx_n \\ &= \sum_{i=1}^{n-1} P_i dX_i + \Theta dX_n, \end{aligned}$$

an equation explicitly free from the variable p_n. Now the quantities $P_1, \ldots\ldots, P_{n-1}, \Theta, X_1, \ldots\ldots, X_n$ are $2n$ functions of $x_1, \ldots\ldots, x_n, p_1, \ldots\ldots,$

p_{n-1}, so that there is one relation between them; and therefore we have the theorem:—

II. *When a single relation exists among 2n quantities* $q_1, \ldots\ldots, q_n$, $y_1, \ldots, y_n$, *then the expression*

$$\sum_{i=1}^{n} q_i dy_i$$

can be transformed to

$$dx_0 + \sum_{i=1}^{n-1} p_i dx_i.$$

(This is an independent derivation of Clebsch's result as to the normal reduced form (§ 127) equivalent to a conditioned differential expression.)

139. Among the cylindrical transformations determined by these theorems there is an important class called *homogeneous*.

If in the equations satisfied by the quantities Π, P, X we take Π to be a constant, say B, then we have

$$\left.\begin{aligned} P_i &= \sum_{r=1}^{n} p_r \frac{\partial P_i}{\partial p_r} \\ 0 &= \sum_{r=1}^{n} p_r \frac{\partial X_i}{\partial p_r} \end{aligned}\right\}$$

for all values of the index i. In this case we therefore infer that the quantities X are homogeneous in the variables p and of zero dimension, and that the quantities P are also homogeneous in the variables p and of one dimension.

Conversely, let the quantities X be homogeneous in the variables p and of zero dimension; then it follows, from the equations

$$(P_1 X_1) = (P_2 X_2) = \ldots\ldots\ldots\ldots = (P_n X_n) = A,$$

that the quantities P are homogeneous in the variables p and of one dimension. Hence we have

$$P_i = \sum_{r=1}^{n} p_r \frac{\partial P_i}{\partial p_r}, \quad 0 = \sum_{r=1}^{n} p_r \frac{\partial X_i}{\partial p_r};$$

and therefore the equations determining Π are

$$(\Pi X_i) = 0, \quad (\Pi P_i) = 0,$$

a set, $2n$ in number, which are linearly independent of one another. Since the conditions for coexistence are satisfied, they form a complete system. But the number of variables is $2n$, the same as the number of the equations; and hence there is no variable solution

(the number of solutions is, by § 38, $2n - 2n$), that is, the only solution is $\Pi = \text{constant}$, say B.

Hence we have the theorem*:—

When $X_1, \ldots, X_n$ are homogeneous functions of zero dimension in the variables p, satisfying the equations

$$(X_i X_j) = 0,$$

then the quantities P are homogeneous of one dimension in the variables p; the value of Z is $Az + B$, where A and B are constants, and the other equations to be satisfied are

$$\left.\begin{array}{l}(P_i P_j) = 0 = (X_i P_j) \\ (P_i X_i) = A\end{array}\right\}.$$

Such a tangential transformation is called homogeneous.

This result attaches itself, as the earlier ones, to Clebsch's investigations on Pfaff's problem, for it is his generalisation of a given normal form equivalent to an unconditioned differential expression. Substituting the value of Z in the characteristic variational relation and absorbing the constant A into the left-hand side, it takes the form

$$\sum_{i=1}^{n} P_i dX_i = \sum_{i=1}^{n} p_i dx_i.$$

Clebsch's equations (§ 122) are

$$(X_i) = 0, \quad (P_i) = P_i,$$

and $[X_i X_j] = 0$, which is in his notation the same as the above

$$(X_i X_j) = 0.$$

The two former equations establish the homogeneous character of the transformation.

Further, since $X_1, \ldots\ldots, X_n$ are homogeneous of no dimension in the variables p, these variables can be eliminated among the quantities X, and the result will be an equation of the form

$$\text{function } (X_1, \ldots\ldots, X_n, x_1, \ldots\ldots, x_n) = 0,$$

which is the foundation of Clebsch's generalisation of a given reduced form.

140. The equations characteristic of a homogeneous transformation, viz.,

$$(X_i X_j) = 0 = (X_i P_j) = (P_i P_j),$$
$$(P_i X_i) = 1,$$

* Lie, *Math. Ann.*, t. viii., p. 238.

supposing the constant A absorbed into the variables X, are satisfied by $X_i = x_i$, $P_i = p_i$: a transformation which is merely identical. But we may have a transformation which causes only an infinitesimal variation*; and its form will be

$$X_i = x_i + \epsilon\xi_i, \quad P_i = p_i + \epsilon\pi_i,$$

where ϵ, an infinitesimal constant, may be taken the same for all. The quantities ξ and π are however not necessarily independent; in fact, as the transformation is used with the retention of small quantities of the first order, ξ and π must be such as to satisfy the characteristic equations within the same limits.

The equation $(X_i X_j) = 0$ leads to the condition

$$\frac{\partial \xi_i}{\partial p_j} = \frac{\partial \xi_j}{\partial p_i};$$

the equation $(P_i P_j) = 0$ to the condition

$$\frac{\partial \pi_i}{\partial x_j} = \frac{\partial \pi_j}{\partial x_i};$$

the equation $(X_i P_j) = 0$, with $i \gtrless j$, to the condition

$$\frac{\partial \pi_j}{\partial p_i} = -\frac{\partial \xi_i}{\partial x_j};$$

and the equation $(P_i X_i) = 1$ to the condition

$$\frac{\partial \pi_i}{\partial p_i} = -\frac{\partial \xi_i}{\partial x_i}.$$

These conditions shew that there exists some function H such that

$$\xi_i = \frac{\partial H}{\partial p_i},$$

$$\pi_i = -\frac{\partial H}{\partial x_i};$$

and, since the transformation is homogeneous, so that ξ is homogeneous and of zero dimension in the quantities p while π is homogeneous and of one dimension in them, it follows that H,

* Such transformations, of the linear or the homographic type, are freely used in the theory of algebraic forms and of classes of functional invariants to establish the partial differential equations characteristic of all invariants.

save as to a negligible additive constant, is homogeneous and of one dimension in the quantities p. Hence we have the theorem*:—

Every cylindrical tangential transformation, which is homogeneous and infinitesimal, is of the form

$$\epsilon = \frac{X_i - x_i}{\dfrac{\partial H}{\partial p_i}} = \frac{P_j - p_j}{-\dfrac{\partial H}{\partial x_j}}, \ (i, j = 1, \ldots, n),$$

where H is a function homogeneous of one dimension in the variables p, and z is left untransformed.

* Lie, *Math. Ann.*, t. viii., p. 240.

CHAPTER X.

LIE'S METHOD.

141. THE reduction of a Pfaffian differential expression to its equivalent normal form is made, in Lie's method, by means of the two theorems proved in § 138, which for convenience may here be repeated:—

I. When a single relation exists among $2n+1$ quantities $y_0, y_1, \ldots, y_n, q_1, \ldots, q_n$, then the expression

$$dy_0 + \sum_{i=1}^{n} q_i dy_i$$

can be transformed to

$$\sum_{i=1}^{n} p_i dx_i.$$

II. When a single relation exists among $2n$ quantities $q_1, \ldots, q_n, y_1, \ldots, y_n$, then the expression

$$\sum_{i=1}^{n} q_i dy_i$$

can be transformed to

$$dx_0 + \sum_{i=1}^{n-1} p_i dx_i.$$

As the transformation merely implies identical equivalence of the transformed expressions, these theorems are still true when more than a single relation exists among the quantities q and y, the difference in the result being that there exists among the quantities p and x a corresponding number of relations.

When these two theorems are applied in alternate succession to a Pfaffian expression

$$\sum_{s=1}^{m} X_s dx_s,$$

where the coefficients X are functions of the variables x, so that relations exist among $X_1, \ldots, X_m, x_1, \ldots, x_m$, the result is that the expression ultimately assumes either the form

$$F_1 df_1 + \ldots\ldots + F_n df_n,$$

where $2n \overline{\overline{<}} m$ and the quantities F and f are completely independent of one another, or the form

$$d\phi_0 + \Phi_1 d\phi_1 + \ldots + \Phi_{n-1} d\phi_{n-1},$$

where $2n - 1 \overline{\overline{<}} m$ and the quantities ϕ and Φ are completely independent of one another.

142. Let the first of these reduced normal forms be called of *even* character (as containing an even number of independent variable quantities) and the second be called of *uneven* character (as containing an odd number of independent variable quantities). Then Lie proves the result, assumed as obvious by Clebsch, that:

All reduced normal forms equivalent to a given differential expression are of the same character in the same number of functions.

There are three cases to be considered.

First, let there be two reduced forms of even character

$$\sum_{i=1}^{n} F_i df_i, \quad \sum_{i=1}^{r} G_i dg_i$$

equivalent to $\sum\limits_{i=1}^{m} X_i dx_i$; then we have identically

$$\sum_{i=1}^{n} F_i df_i - \sum_{i=1}^{r} G_i dg_i = 0,$$

an equation involving $n + r$ differential elements. An identical equation of this type, in which the coefficients of the differential elements do not vanish, can be satisfied only by means of $n + r$ relations connecting the quantities F, f, G, g. Of the two numbers, suppose r not the greater; then eliminating from the $n + r$ relations the $2r$ quantities G and g, there are $n - r$ relations left among the quantities F and f. These quantities are independent, because they occur in the reduced normal form; hence $n - r = 0$, or the two

normal forms of even character equivalent to a differential expression contain the same number of functions *.

Secondly, let there be two reduced forms of uneven character

$$df_0 + \sum_{i=1}^{n} F_i df_i, \quad dg_0 + \sum_{i=1}^{r} G_i dg_i$$

equivalent to a differential expression; then we have identically

$$d(f_0 - g_0) + \sum_{i=1}^{n} F_i df_i - \sum_{i=1}^{r} G_i dg_i = 0.$$

This identical equation is of the same type as before and can therefore be satisfied only by means of $n + r + 1$ relations among the quantities $f_0 - g_0$, F_i, f_i, G_i, g_i. Taking r to be that one of the two numbers which is not the greater, we have after the elimination of the $2r + 1$ quantities $f_0 - g_0$, G and g from these relations, $n - r$ relations left among the quantities F_i and f_i. But these quantities are independent; hence $n - r = 0$, or the two

* The question as to the number of independent integral equations, in virtue of which an irreducible differential expression

$$v_1 du_1 + v_2 du_2 + \ldots\ldots + v_n du_n$$

can vanish identically, has already (§ 69) been partially considered.

Every member of the integral system will belong to one or other of three classes: (i) equations involving the variables u alone, (ii) equations involving the variables v alone, (iii) equations involving the variables u and v. Let the integral system be so modified that no equation of the first class can be derived by means of the equations of the second and third, in fact, so that the variables v cannot be eliminated from among the equations of those classes: a little consideration will shew that any functional combination of the equations of the integral system is permissible.

Let there be μ equations of the first class, ρ of the second and σ of the third; then the number of variables v occurring in the latter $\rho + \sigma$ equations must be $\geqq \rho + \sigma$, for otherwise elimination could take place. Evidently $\mu \leqq n$.

When the $\rho + \sigma$ equations are differentiated, they lead to $\rho + \sigma$ relations among the differential elements du and dv. Since the number of the elements dv is either equal to or greater than this number of relations, no relation or set of relations involving the elements du alone can be derived from them. Hence the only equations which can lead to relations among the differential elements du are those of the first class, μ in number.

If $\mu = n$, then, since the equations are independent, the quantities u are determinate constants, so that

$$du_1 = 0, \; du_2 = 0, \; \ldots\ldots, \; du_n = 0,$$

and therefore the equation

$$v_1 du_1 + v_2 du_2 + \ldots\ldots + v_n du_n = 0 \quad \text{(i)}$$

is identically satisfied by means of the n equations

$$u_1 = \text{constant}, \; \ldots\ldots, \; u_n = \text{constant}.$$

normal forms of uneven character equivalent to a differential expression contain the same number of functions.

Thirdly, two reduced forms equivalent to a differential expression must be of the same character. For, if otherwise, let there be two reduced forms

$$\sum_{i=1}^{n} F_i df_i, \quad d\phi_0 + \sum_{i=1}^{r} \Phi_i d\phi_i$$

equivalent to the same expression; then

$$d\phi_0 + \sum_{i=1}^{r} \Phi_i d\phi_i - \sum_{i=1}^{n} F_i df_i = 0$$

is an identical equation, which can be satisfied only by means of $n+r+1$ relations among the quantities ϕ, Φ, f, F. If r be $\geqq n$, this number of relations is greater than $2n$, so that the elimination among them of the $2n$ quantities F and f will leave relations among the quantities Φ and ϕ, contrary to possibility; while, if r be $< n$, the number of relations is greater than $2r+1$,

If $\mu < n$, then the μ independent equations can be solved so as to express μ of the quantities u in terms of the remaining $n-\mu$, say in the form

$$u_i = f_i(u_{\mu+1}, \ldots\ldots, u_n),$$

where $i = 1, 2, \ldots\ldots, \mu$. Substituting for the quantities u_i, we have (i) replaced by

$$\sum_{\rho=\mu+1}^{n} \left\{ \sum_{i=1}^{\mu} v_i \frac{\partial f_i}{\partial u_\rho} + v_\rho \right\} du_\rho = 0.$$

There are no further relations among the quantities u_ρ and therefore no relations among the elements du_ρ; hence the modified equation can be satisfied identically only by the equations

$$\sum_{i=1}^{\mu} v_i \frac{\partial f_i}{\partial u_\rho} + v_\rho = 0$$

for $\rho = \mu+1, \ldots\ldots, n$. These are $n-\mu$ in number; combined with the former μ equations, they constitute an aggregate of n equations.

For the most general forms of the functions f_i, all the $n-\mu$ derived equations fall into the third class of equations mentioned; but a member of the second class will occur for a value of ρ when all the functions f_i either are completely independent of u_ρ, or involve u_ρ only in an additive term which is linear in u_ρ, or involve u_ρ only as a linear factor.

The general result therefore is:—

An irreducible expression

$$v_1 du_1 + v_2 du_2 + \ldots\ldots + v_n du_n$$

can be made to vanish identically only in virtue of a system of n integral equations.

See Gauss, *Ges. Werke*, t. iii., p. 235; Grassmann's *Ausdehnungslehre* (1862), p. 352; Lie (l.c., p. 343).

so that the elimination among them of the $2r+1$ quantities ϕ and Φ will leave relations among the quantities F and f, contrary to possibility. Hence the inferred identical equation cannot exist; and therefore the two reduced normal forms of different character cannot be equivalent to the same expression.

Ex. The following theorem, closely connected with what precedes, can easily be proved:

If two differential expressions

$$\sum_{i=1}^{m} X_i dx_i, \quad \sum_{i=1}^{n} Y_i dy_i$$

(each in any number of variables) have normal forms, which are of the same character and contain the same number of functions, then the expressions can be transformed into one another.

Lie's proof assumes the limitation $m=n$; the modification to this more general form is easily effected.

143. The equations connecting the functions in two reduced normal forms, equivalent to a given differential expression, are derived by Lie from the results of the theory of tangential transformations; they are, of course, similar to the corresponding equations in Clebsch's theory.

If there be two such forms of even character, say

$$\sum_{i=1}^{n} F_i df_i, \quad \sum_{i=1}^{n} G_i dg_i,$$

equivalent to the same expression so that

$$\sum_{i=1}^{n} G_i dg_i = \sum_{i=1}^{n} F_i df_i,$$

where all the quantities on the same side of the equation are independent of one another, then we may apply the results of the homogeneous tangential transformation (§ 139). Taking

$$(UV) = \sum_{r=1}^{n} \left(\frac{\partial U}{\partial F_r} \frac{\partial V}{\partial f_r} - \frac{\partial U}{\partial f_r} \frac{\partial V}{\partial F_r} \right),$$

then the equations satisfied by the quantities G and g are

$$\left.\begin{aligned} &(g_i,\ g_j) = 0, \quad \sum_{r=1}^{n} F_r \frac{\partial g_i}{\partial F_r} = 0 \\ &(g_i,\ G_j) = 0 = (G_i,\ G_j) \\ &(G_i,\ g_i) = 1, \quad \sum_{r=1}^{n} F_r \frac{\partial G_i}{\partial F_r} = G_i \end{aligned}\right\} .$$

It is sufficient to take, for the quantities g, such homogeneous functions of the variables F of zero dimension as satisfy the equations

$$(g_i,\ g_j) = 0;$$

and some one of the functions, say g_1, may be taken an arbitrary function of the variables f and F subject only to the limitation of homogeneity.

If there be two reduced forms of uneven character, say

$$d\phi_0 + \sum_{i=1}^{n} \Phi_i d\phi_i, \quad d\theta_0 + \sum_{i=1}^{n} \Theta_i d\theta_i,$$

equivalent to the same expression so that

$$d\theta_0 + \sum_{i=1}^{n} \Theta_i d\theta_i = d\phi_0 + \sum_{i=1}^{n} \Phi_i d\phi_i,$$

where all the quantities on the same side of the equation are independent of one another, the results of the cylindrical tangential transformation (§ 136) may be applied. From those results, we have

$$\theta_0 = \phi_0 - \Pi,$$

where Π is a function of the quantities $\phi_1, \ldots, \phi_n, \Phi_1, \ldots, \Phi_n$ alone; and the equations satisfied by Π and the quantities $\theta_1, \ldots, \theta_n$ are

$$(\theta_i,\ \theta_j) = 0,\ (\Pi\theta_i) = \sum_{r=1}^{n} \Phi_r \frac{\partial \theta_i}{\partial \Phi_r},$$

where

$$(UV) = \sum_{r=1}^{n} \left(\frac{\partial U}{\partial \Phi_r}\frac{\partial V}{\partial \phi_r} - \frac{\partial U}{\partial \phi_r}\frac{\partial V}{\partial \Phi_r}\right);$$

and the quantities $\Theta_1, \ldots, \Theta_n$ are given by any n independent equations of the set

$$\frac{\partial \Pi}{\partial \phi_i} - \sum_{r=1}^{n} \Theta_r \frac{\partial \theta_r}{\partial \phi_i} = -\Phi_i, \quad \frac{\partial \Pi}{\partial \Phi_i} - \sum_{r=1}^{n} \Theta_r \frac{\partial \theta_r}{\partial \Phi_i} = 0.$$

144. It is important to determine *a priori* the number of functions which occur in the normal form equivalent to a given expression

$$\sum_{i=1}^{n} X_i dx_i.$$

In accordance with § 141, this may be reduced to a form of even or to one of uneven character.

If it have been reduced to one of even character, say to

$$\sum_{i=1}^{n} F_i df_i,$$

where $2n \leqq m$, then it cannot be further reduced, unless some functional relation subsists among the quantities F and f. Let $\alpha, \beta, \ldots, \rho$ be any $2n$ integers of the series $1, 2, \ldots, m$; then since the Jacobian

$$\frac{\partial(f_1, \ldots, f_n, F_1, \ldots, F_n)}{\partial(x_\alpha, x_\beta, \ldots\ldots\ldots\ldots, x_\rho)}$$

is, save as to sign, equal to the Pfaffian $[\alpha, \beta, \ldots, \rho]$, it follows that the form $\sum_{i=1}^{n} F_i df_i$ can or cannot be further reduced, according as all the Pfaffians $[\alpha, \beta, \ldots, \rho]$ do or do not vanish.

If the given expression have been reduced to a form of uneven character, say to

$$d\phi_0 + \sum_{i=1}^{n} \Phi_i d\phi_i,$$

where $2n+1 \leq m$, then it cannot be further reduced unless some functional relation subsist among the quantities ϕ and Φ. Let $a, b, \ldots, i, \ldots, l$ be any $2n+1$ integers from the series $1, 2, \ldots, m$: then the quantities ϕ and Φ are or are not independent, according as all the Jacobians

$$\nabla = \frac{\partial(\phi_0, \phi_1, \ldots, \phi_n, \Phi_1, \ldots, \Phi_n)}{\partial(x_a, x_b, \ldots\ldots\ldots\ldots\ldots\ldots, x_l)}$$

do not or do vanish. Now this Jacobian is equal to

$$\sum_{i=a, b, \ldots, l} \pm \frac{\partial\phi_0}{\partial x_i} \frac{\partial(\phi_1, \ldots, \phi_n, \Phi_1, \ldots, \Phi_n)}{\partial(\quad \ldots x_l, x_a, x_b, \ldots \quad)},$$

x_i being omitted in the new Jacobian ∇_i which multiplies $\dfrac{\partial\phi_0}{\partial x_i}$: and hence

$$\nabla = \Sigma X_i \nabla_i.$$

As before, ∇_i is the Pfaffian $[\ldots, l, a, b, \ldots]$ of order $2n$, save as to a sign which is the same for all the quantities ∇_i: thus

$$\nabla = \pm \sum_{i=a, b, \ldots, l} X_i [\ldots, l, a, b, \ldots],$$

where the symbol of the Pfaffian begins with the integer in the series $a, b, \ldots, l$ next after i and does not contain i.

If all the quantities

$$\sum_{i=a, b, \ldots, l} X_i [\ldots, l, a, b, \ldots]$$

do not vanish, the form cannot be further reduced: while, if they all do vanish, a reduction can be effected.

Combining these two results, we have the following conditions determining the number of functions in the reduced normal form:—

If all the Pfaffians, constructed from the coefficients of the differential expression, of orders higher than $2n$ *but not all those of order* $2n$ *vanish, then the reduced normal form contains either* $2n$ *or* $2n+1$ *functions. It is of even or of uneven character, that is, it contains* $2n$ *or* $2n+1$ *functions, according as all the expressions*

$$\sum_{i=a, b, \ldots, l} X_i [\ldots, l, a, b, \ldots],$$

for every combination of $2n+1$ *integers* $a, b, \ldots, l$ *from the series* $1, \ldots, m$, *do or do not vanish**.

145. Suppose then that a differential expression in $2n+q$ variables

$$\Omega_{2n+q} = \sum_{i=1}^{2n+q} X_i dx_i$$

has a reduced normal form of the character

$$\sum_{r=1}^{n} F_r df_r,$$

containing $2n$ functions, necessarily independent. Now all the Pfaffians of order $2n$ do not vanish: let a non-vanishing Pfaffian be $[1, \ldots, 2n]$, that is, one in which the variables of differentiation are $x_1, \ldots, x_{2n}$. Then there is no relation among $f_1, \ldots, f_n$, $F_1, \ldots, F_n, x_{2n+1}, \ldots, x_{2n+q}$; for the non-evanescent Pfaffian just mentioned is equal to the Jacobian of $f_1, \ldots, f_n, F_1, \ldots, F_n$ with

* This result is not enunciated by Lie, but it can be inferred from his investigations: and it is more explicit and definite than the gradual process indicated by him (l.c., § 3, p. 352).

regard to $x_1, \ldots, x_{2n}$. Further, at least two of the quantities W of § 59 do not vanish; let one of them be W_{2n}, so that the quantity

$$W_{2n} = X_1[2, \ldots, 2n-1] + X_2[3, \ldots, 2n-1, 1] + \ldots \\ + X_{2n-1}[1, \ldots, 2n-2]$$

is different from zero. Now

$$[2, \ldots, 2n-1] = \Sigma \frac{\partial(F_a, f_a, F_b, f_b, \ldots, F_k, f_k)}{\partial(x_2, x_3, \ldots\ldots\ldots\ldots, x_{2n-1})},$$

where $a, b, \ldots, k$ are $n-1$ integers of the series $1, 2, \ldots, n$, and the summation on the right-hand side extends to the n terms which correspond to the n ways in which those integers can be chosen. Also

$$X_i = \sum_{r=1}^{n} F_r \frac{\partial f_r}{\partial x_i};$$

hence, substituting in W_{2n}, we have

$$\begin{aligned} W_{2n} = F_1 & \frac{\partial(f_1, F_2, f_2, F_3, f_3, \ldots, F_n, f_n)}{\partial(x_1, x_2, \ldots\ldots\ldots\ldots\ldots\ldots, x_{2n-1})} \\ & + F_2 \frac{\partial(f_2, F_3, f_3, \ldots, F_n, f_n, F_1, f_1)}{\partial(x_1, x_2, \ldots\ldots\ldots\ldots\ldots\ldots, x_{2n-1})} + \ldots \\ & \ldots + F_n \frac{\partial(f_n, F_1, f_1, \ldots, F_{n-1}, f_{n-1})}{\partial(x_1, x_2, \ldots\ldots\ldots\ldots\ldots, x_{2n-1})} \\ = F_n^n & \frac{\partial\left(f_1, f_2, \ldots, f_n, \dfrac{F_1}{F_n}, \dfrac{F_2}{F_n}, \ldots, \dfrac{F_{n-1}}{F_n}\right)}{\partial(x_1, x_2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots, x_{2n-1})}. \end{aligned}$$

Since W_{2n} does not vanish, the Jacobian does not vanish; and therefore the quantities

$$f_1, f_2, \ldots, f_n, \frac{F_1}{F_n}, \ldots, \frac{F_{n-1}}{F_n}, x_{2n}, x_{2n+1}, \ldots, x_{2n+q}$$

are independent of one another.

Similarly, if the differential expression in $2n+q$ variables have a reduced normal form of the character

$$\phi_0 + \sum_{r=1}^{n} \Phi_r \, d\phi_r$$

containing $2n+1$ independent functions, then not all the expressions

$$\sum_{i=a, b, \ldots, l} X_i[\ldots, l, a, b, \ldots]$$

vanish; let a non-vanishing expression be

$$X_1[2, \ldots, 2n+1]+X_2[3, \ldots, 2n+1, 1]+\ldots+X_{2n+1}[1, \ldots, 2n].$$

Further, not all the Pfaffians of order $2n$ can vanish, otherwise the above expression would vanish; let such a non-vanishing Pfaffian be $[1, \ldots, 2n]$. Since this is equal to the Jacobian of $\phi_1, \ldots, \phi_n$, $\Phi_1, \ldots, \Phi_n$, it follows that the quantities

$$\phi_1, \phi_2, \ldots, \phi_n, \Phi_1, \ldots, \Phi_n, x_{2n+1}, \ldots, x_{2n+q}$$

are independent of one another.

146. The process adopted by Lie for the construction of the normal form is composed of three distinct kinds of operations. The first is the modification of the given differential expression, the character of whose normal form has been inferred by the tests of § 144, so that it becomes an unreduced and unconditioned expression with the smallest number of differential elements consistent with the existence of such a form. The second is the construction of the normal form of that modified expression. The third is the transition from that normal form to the normal form of the original expression. The construction is gradual; and the different operations are applied in alternating succession in that gradual reduction.

The method, which is most nearly akin to Lie's, is Clebsch's first method (§§ 117—119) when it is combined with Mayer's method of solution of the systems of partial differential equations which occur therein. The characteristic variation in the present method is the application to the differential expression itself of the Cauchy substitutions before any of the subsidiary equations are constructed.

147. Suppose then that the differential expression

$$\Omega = \sum_{i=1}^{2n+q} X_i\, dx_i$$

in $2n+q$ variables is recognised by the tests of § 144 to have a normal form of even character involving $2n$ functions, say

$$\Omega = \sum_{i=1}^{n} F_i df_i,$$

where the functions f and F are independent of one another.

Then a non-vanishing Pfaffian (§ 145) of order $2n$ may be taken to be $[1, 2, \ldots, 2n]$, and also an expression

$$X_1[2, \ldots, 2n-1] + X_2[3, \ldots, 2n-1, 1] + \ldots\ldots + X_{2n-1}[1, \ldots, 2n-2]$$

may be supposed not to vanish, thus implying that the quantities

$$f_1, \ldots, f_n, \frac{F_1}{F_n}, \ldots, \frac{F_{n-1}}{F_n}, x_{2n}, x_{2n+1}, \ldots, x_{2n+q}$$

are independent of one another.

To change Ω into an expression containing $2n$ differential elements, we make the substitution

$$x_{2n+k} = a_{2n+k} + (x_{2n} - a_{2n})\, y_k$$

for $k = 1, 2, \ldots, q$; and we consider the quantities y_k as invariable*. If X_μ, after the above substitutions have been made, be denoted by Y_μ, we have Ω modified to Ω'_{2n}, where

$$\Omega'_{2n} = \sum_{i=1}^{2n-1} Y_i dx_i + (Y_{2n} + y_1 Y_{2n+1} + \ldots + y_q Y_{2n+q}) dx_{2n}.$$

Then the *normal form of* Ω'_{2n} *contains* $2n$ *functions.* For, since the Pfaffian $[1, \ldots, 2n]$ associated with Ω does not vanish and is the Jacobian of $f_1, \ldots, F_n$ with respect to $x_1, \ldots, x_{2n}$, the new Pfaffian $[1, \ldots, 2n]'$ associated with Ω'_{2n} must be the Jacobian with respect to $x_1, \ldots, x_{2n}$ of the modified functions $f_1, \ldots, F_n$, say of $f_1^{(y)}, \ldots, F_n^{(y)}$. Now this new Jacobian may not vanish: for there would then be a relation among $f_1^{(y)}, \ldots, F_n^{(y)}$ involving (it may be) the quantities y and a. When in the quantities $f_1^{(y)}, \ldots, F_n^{(y)}$ we substitute for y, they change into $f_1, \ldots, F_n$ respectively, so that the inferred relation changes into one between $f_1, \ldots, F_n$ and a number of arbitrarily assumed constants a and y, but independent of $x_1, \ldots, x_{2n}$. This result is excluded by the supposition that $\Sigma F df$ is the normal form of Ω; and therefore there is no relation among the quantities $f^{(y)}$ and $F^{(y)}$, or the Pfaffian of order $2n$ of Ω'_{2n} does not vanish. Hence the normal form of Ω'_{2n} contains $2n$ functions. Thus:—

The substitutions

$$x_{2n+k} = a_{2n+k} + (x_{2n} - a_{2n})\, y_k,$$

* Lie uses the symbol λ instead of y; the above notation is used, in accordance with Mayer's (§§ 35, 41), the first substitution (which is merely identical viz., $x_{2n} = y$) being omitted.

for $k=1, 2, \ldots, q$; *change* Ω *into an expression* Ω'_{2n}, *the normal form of which contains* $2n$ *functions.*

The importance of the transformation is due to the fact that from the normal form of Ω'_{2n} we can deduce the normal form of Ω merely by algebraical operations: to the proof of this we now proceed.

148. Let it be supposed, then, that a normal form of Ω'_{2n} is

$$\Omega'_{2n} = \Phi_1 d\phi_1 + \ldots\ldots + \Phi_n d\phi_n,$$

where the quantities ϕ and Φ are $2n$ independent functions of the variables $x_1, \ldots, x_{2n}$ and of the quantities y and α: and denote the normal form of Ω by

$$\Omega = F_1 df_1 + \ldots\ldots + F_n df_n.$$

When the substitutions for the variables are made in Ω and in $\Sigma F df$, it follows that $\Sigma F^{(y)} df^{(y)}$ is a normal form of Ω'_{2n} and thus that $\phi_1, \ldots, \phi_n, \dfrac{\Phi_1}{\Phi_n}, \ldots, \dfrac{\Phi_{n-1}}{\Phi_n}$ are $2n-1$ independent functions of $f_1^{(y)}, \ldots, f_n^{(y)}, \dfrac{F_1^{(y)}}{F_n^{(y)}}, \ldots, \dfrac{F_{n-1}^{(y)}}{F_n^{(y)}}$.

Denoting F_μ/F_n by $f_{n+\mu}$ for $\mu=1, 2, \ldots, n-1$, we have the quantities

$$f_1, \ldots, f_n, f_{n+1}, \ldots, f_{2n-1}, x_{2n}, \ldots, x_{2n+q}$$

independent of one another; and therefore the system of equations

$$A_k f = \frac{\partial(f_1, \ldots\ldots, f_{2n-1}, f)}{\partial(x_1, \ldots\ldots, x_{2n-1}, x_{2n+k})} = 0$$

for $k=0, 1, \ldots, q$ possess the $2n-1$ independent solutions $f=f_1, f_2, \ldots, f_{2n-1}$. Moreover $\dfrac{\partial f}{\partial x_{2n+k}}$ occurs only in $A_k f$; the equations are therefore algebraically independent of one another.

These equations are linear and homogeneous in the partial differential coefficients of f and they do not involve the dependent variable f itself: they are thus of the type to which Mayer's method may be applied. The system contains $q+1$ equations and the number of variables occurring is $2n+q$; further, it is known that there are $2n-1\ (= 2n+q-q-1)$ independent solutions. Hence the system is a complete system.

The equation

$$A_k f = 0$$

may be written

$$\Theta \frac{\partial f}{\partial x_{2n+k}} + \sum_{s=1}^{2n-1} U_{k,s} \frac{\partial f}{\partial x_s} = 0,$$

where

$$U_{k,s} = \frac{\partial (f_1, \ldots, f_{2n-1})}{\partial_s (x_1, \ldots, x_{2n-1}, x_{2n+k})},$$

the subscript in ∂_s implying that x_s is omitted from the series of variables x in the determinant, and

$$\Theta = \frac{\partial (f_1, \ldots, f_{2n-1})}{\partial (x_1, \ldots, x_{2n-1})}$$

being the same for all the equations. When Mayer's method is applied with the simplest substitutions (§ 35), the single equation which serves to determine an integral system is

$$\Theta^{(y)} \frac{\partial f}{\partial x_{2n}} + \sum_{s=1}^{2n-1} Y_s \frac{\partial f}{\partial x_s} = 0,$$

where $\Theta^{(y)}$ is the value of Θ after these substitutions are made in Θ, and the coefficients Y are given by the equations

$$\begin{aligned} Y_s &= U_{0s} + y_1 U_{1s} + y_2 U_{2s} + \ldots\ldots + y_q U_{qs} \\ &= \left| \frac{\partial (f_1, \ldots, f_{2n-1})}{\partial_s (x_1, \ldots, x_{2n})} \right|^{(y)} + y_1 \left| \frac{\partial (f_1, \ldots, f_{2n-1})}{\partial_s (x_1, \ldots, x_{2n+1})} \right|^{(y)} + \ldots\ldots \\ &\qquad + y_q \left| \frac{\partial (f_1, \ldots, f_{2n-1})}{\partial_s (x_1, \ldots, x_{2n+q})} \right|^{(y)}. \end{aligned}$$

But

$$\left| \frac{\partial \theta}{\partial x_\mu} \right|^{(y)} = \frac{\partial \theta^{(y)}}{\partial x_\mu} \text{ for } \mu = 1, \ldots, 2n-1;$$

and

$$\frac{\partial \theta^{(y)}}{\partial x_{2n}} = \left| \frac{\partial \theta}{\partial x_{2n}} \right|^{(y)} + y_1 \left| \frac{\partial \theta}{\partial x_{2n+1}} \right|^{(y)} + \ldots\ldots + y_q \left| \frac{\partial \theta}{\partial x_{2n+q}} \right|^{(y)}.$$

Hence we have

$$Y_s = \frac{\partial \left(f_1^{(y)}, \ldots, f_{2n-1}^{(y)} \right)}{\partial_s (x_1, \ldots, x_{2n})},$$

and

$$\Theta^{(y)} = \frac{\partial \left(f_1^{(y)}, \ldots, f_{2n-1}^{(y)} \right)}{\partial (x_1, \ldots\ldots, x_{2n-1})},$$

so that the single subsidiary Mayer-equation is

$$\frac{\partial\left(f_1^{(y)}, \ldots, f_{2n-1}^{(y)}, f\right)}{\partial(x_1, \ldots\ldots, x_{2n-1}, x_{2n})} = 0,$$

an equation in $2n$ variables and having, therefore, $2n-1$ independent solutions. One evident system of solutions is $f_1^{(y)}, \ldots, f_{2n-1}^{(y)}$; any system of $2n-1$ independent functions of these quantities is also a system of solutions. Such a set of independent functions is constituted by $\phi_1, \ldots, \phi_n, \frac{\Phi_1}{\Phi_n}, \ldots, \frac{\Phi_{n-1}}{\Phi_n}$, a set of functions already supposed to be known because a normal form of Ω'_{2n} is supposed known.

It thus appears that a full integral-system of the subsidiary equation is known.

Then, by Mayer's theorem (§ 41), one system of integrals of the system of equations $A_k f = 0$ is given by forming the equations

$$\phi_\mu(x_1, \ldots, x_{2n-1}, x_{2n}, y_1, \ldots, y_q) = \phi_\mu(h_1, \ldots, h_{2n-1}, a_{2n}, y_1, \ldots, y_q)$$

for $\mu = 1, 2, \ldots, 2n-1$, the quantity $\phi_{n+\lambda}$ denoting Φ_λ/Φ_n. In these equations the variables y are replaced by their values in terms of the variables x; the $2n-1$ equations are solved for the $2n-1$ quantities h with results of the form

$$h_\mu = h_\mu(x_1, \ldots, x_{2n+q}),$$

where the function on the right-hand side becomes x_μ by the substitutions $x_{2n+k} = a_{2n+k}$ for $k = 0, 1, \ldots, q$. These are the principal integrals of the system of equations $A_k f = 0$; and the functions $f_1, \ldots, f_{2n-1}$ are $2n-1$ independent functions of these principal integrals.

149. We now pass to the construction of a normal form of Ω. We have

$$\Omega = \sum_{i=1}^{2n+q} X_i dx_i = (f_{n+1} df_1 + f_{n+2} df_2 + \ldots + f_{2n-1} df_{n-1} + df_n) F_n;$$

substitution on the right-hand side for the functions $f_1, \ldots, f_{2n-1}$, in terms of the principal integrals h, leads to a result of the form

$$\Omega = \rho \sum_{i=1}^{2n-1} H_i(h_1, \ldots, h_{2n-1})\, dh_i,$$

where ρ and the functions H_i are unknown. In this equation we make $x_{2\mu+\mu} = a_{2n+\mu}$ for $\mu = 0, 1, 2, \ldots, q$; then $h_i = x_i$, so that we have

$$\sum_{i=1}^{2n-1} X_i(x_1, \ldots, x_{2n-1}, a_{2n}, \ldots, a_{2n+q})\, dx_i = \rho_a \sum_{i=1}^{2n-1} H_i(x_1, \ldots, x_{2n-1})\, dx_i.$$

This is merely an identity, so that, if σ_a denote ρ_a when the quantities x are replaced by the quantities h, we have

$$\sum_{i=1}^{2n-1} X_i(h_1, \ldots, h_{2n-1}, a_{2n}, \ldots, a_{2n+q})\, dh_i = \sigma_a \sum_{i=1}^{2n-1} H_i(h_1, \ldots, h_{2n-1})\, dh_i,$$

and thence

$$\Omega = \frac{\sigma_a}{\rho} \sum_{i=1}^{2n-1} X_i(h_1, \ldots, h_{2n-1}, a_{2n}, \ldots, a_{2n+q})\, dh_i.$$

Returning to the modified form of Ω, we have

$$\Omega'_{2n} = \sum_{i=1}^{n} \Phi_i(x_1, \ldots, x_{2n-1}, x_{2n}, y_1, \ldots, y_q)\, d\phi_i(x_1, \ldots, x_{2n-1}, x_{2n}, y_1, \ldots, y_q);$$

taking $x_{2n} = a_{2n}$ so that $x_{2n+j} = a_{2n+j}$, we have, as the new form of Ω'_{2n}, $\sum_{i=1}^{2n-1} X_i(x_1, \ldots, x_{2n-1}, a_{2n}, \ldots, a_{2n+q})\, dx_i$, and therefore

$$\sum_{i=1}^{2n-1} X_i(x_1, \ldots, x_{2n-1}, a_{2n}, \ldots, a_{2n+q})\, dx_i$$

$$= \sum_{i=1}^{n} \Phi_i(x_1, \ldots, x_{2n-1}, a_{2n}, y_1, \ldots, y_q)\, d\phi_i(x_1, \ldots, x_{2n-1}, a_{2n}, y_1, \ldots, y_q).$$

Since this is merely an identity, we have also

$$\sum_{i=1}^{2n-1} X_i(h_1, \ldots, h_{2n-1}, a_{2n}, \ldots, a_{2n+q})\, dh_i$$

$$= \sum_{i=1}^{n} \Phi_i(h_1, \ldots, h_{2n-1}, a_{2n}, y_1, \ldots, y_q)\, d\phi_i(h_1, \ldots, h_{2n-1}, a_{2n}, y_1, \ldots, y_q),$$

and therefore

$$\Omega = \frac{\sigma_a}{\rho} \sum_{i=1}^{n} \Phi_i(h_1, \ldots, h_{2n-1}, a_{2n}, y_1, \ldots, y_q)\, d\phi_i(h_1, \ldots, h_{2n-1}, a_{2n}, y_1, \ldots, y_q),$$

which is a normal form of the original expression.

The value of σ_a/ρ is given by any one of the equations

$$X_\lambda = \frac{\sigma_a}{\rho} \sum_{i=1}^{n} \Phi_i(h_1, \ldots, h_{2n-1}, a_{2n}, y_1, \ldots, y_q) \frac{\partial}{\partial x_\lambda} \phi_i(h_1, \ldots, h_{2n-1}, a_{2n}, y_1, \ldots, y_q)$$

for $\lambda = 1, 2, \ldots, 2n+q$.

Since the quantities $h_1, \ldots, h_{2n-1}$ are derivable from ϕ and Φ by merely algebraical operations, it follows that the above given normal form of Ω is determinately derived from the normal form of the modified expression Ω'_{2n}. Hence the construction of the normal form of even character of a conditioned differential expression depends upon the normal form of an unconditioned differential expression in accordance with the following theorem, due to Lie:—

When a differential expression in $2n+q$ variables

$$\Omega = \sum_{i=1}^{2n+q} X_i dx_i$$

has a normal form of even character in $2n$ functions, the substitutions

$$x_{2n+k} = a_{2n+k} + (x_{2n} - a_{2n})\, y_k,$$

where the quantities y are invariable and $k = 1, \ldots, q$, transform Ω into an expression Ω'_{2n} with a similar normal form. If this normal form of Ω'_{2n} be

$$\sum_{i=1}^{n} \Phi_i d\phi_i,$$

where ϕ and Φ are functions of the variables x and of the quantities a and y, then the normal form of Ω is

$$\omega \sum_{i=1}^{n} \Phi_i (h_1, \ldots, h_{2n-1}, a_{2n}, y_1, \ldots, y_q)\, d\phi_i (h_1, \ldots, h_{2n-1}, a_{2n}, y_1, \ldots, y_q),$$

where ω is determined by any one of the equations

$$X_r = \sum_{i=1}^{n} \Phi_i (h, a, y) \frac{\partial}{\partial x_r} \phi_i (h, a, y),$$

and the $2n-1$ quantities h, being independent functions of the variables and of a and y, are determined by the $2n-1$ equations

$$\vartheta (x_1, \ldots, x_{2n-1}, x_{2n}, y_1, \ldots, y_q) = \vartheta (h_1, \ldots, h_{2n-1}, a_{2n}, y_1, \ldots, y_q),$$

the functional symbol ϑ denoting $\phi_1, \ldots, \phi_n, \Phi_1/\Phi_n, \ldots, \Phi_{n-1}/\Phi_n$ in turn. And then a retransformation, by substituting for y in terms of the variables x, leads to an explicit expression for the normal form of Ω.

Ex. 1. The tests of § 144 shew that the expression

$$(x_1x_2 + x_2x_6)\, dx_1 + (x_1^2 + x_2x_5)\, dx_2 + x_1x_4dx_3 + x_1x_3dx_4 + (x_1x_6 + x_2^2)\, dx_5 + (x_1x_5 + x_1x_2)\, dx_6 = \Omega$$

has a normal form of the general type $Fdf + Gdg$ We have the Pfaffian [1256] not zero, as well as the expression

$$[12]\,X_5 + [25]\,X_1 + [51]\,X_2;$$

and therefore $f, g, \frac{G}{F}, x_6, x_3, x_4$ are independent.

Substituting

$$x_3 = a + y\,(x_6 - \epsilon),\quad x_4 = \beta + z\,(x_6 - \epsilon)$$

in Ω (with y and z invariable), the new form is

$$\Omega' = (x_1x_2 + x_2x_6)\,dx_1 + (x_1^2 + x_2x_5)\,dx_2 + (x_1x_6 + x_2^2)\,dx_5 \\ + (x_1x_5 + x_1x_2 + x_1x_4y + x_1x_3z)\,dx_6.$$

A normal form of Ω' is

$$x_1d\phi_1 + x_2d\phi_2,$$

where

$$\phi_2 = x_1x_6 + x_2x_5,$$
$$\phi_1 = x_1x_2 + x_5x_6 + (y\beta + za)\,x_6 + yz\,(x_6^2 - 2\epsilon x_6).$$

Hence we take

$$x_1x_2 + x_5x_6 + (y\beta + za)\,x_6 + yz\,(x_6^2 - 2\epsilon x_6) = h_1h_2 + h_5\epsilon + (y\beta + za)\,\epsilon - yz\epsilon^2,$$
$$x_1x_6 + x_2x_5 = h_2h_5 + h_1\epsilon,$$
$$\frac{x_2}{x_1} = \frac{h_2}{h_1}.$$

From these we have

$$\frac{\Phi_2(h_1, \ldots\ldots)}{\Phi_1(h_1, \ldots\ldots)} = \frac{h_2}{h_1} = \frac{x_2}{x_1};$$
$$\phi_2(h_1, \ldots\ldots) = x_1x_6 + x_2x_5;$$
$$\phi_1(h_1, \ldots\ldots) = x_1x_2 + x_3x_4 + x_5x_6 - a_3a_4;$$

where a_3 and a_4 are the values of x_3 and x_4 respectively when $x_6 = 0$.

Now

$$\Omega = \omega\,\{\Phi_1(h_1, \ldots\ldots)\,d\phi_1(h_1, \ldots\ldots) + \Phi_2(h_1, \ldots\ldots)\,d\phi_2(h_1, \ldots\ldots)\}$$
$$= \omega'\left[d\,(x_1x_2 + x_3x_4 + x_5x_6 - a_3a_4) + \frac{x_2}{x_1}\,d\,(x_1x_6 + x_2x_5)\right],$$

where ω' denotes $\omega\Phi_1(h_1, \ldots\ldots)$. To find ω' we have

$$x_1x_2 + x_2x_6 = X_1$$
$$= \omega'\left(x_2 + \frac{x_2}{x_1}x_6\right),$$

so that $\omega' = x_1$. Also the constant a_3a_4 may be dropped, on account of the differentiation; and therefore a normal form of Ω is

$$x_1d\,(x_1x_2 + x_3x_4 + x_5x_6) + x_2d\,(x_1x_6 + x_2x_5).$$

All equivalent normal forms can be obtained by the equations of § 143.

Ex. 2. Discuss similarly the expression $\sum_{i=1}^{6} X_i dx_i$, where

$$X_1 = x_1 x_6^2 + x_2 x_5 (x_2 + x_4),\quad X_2 = x_1 x_5 (x_2 + x_4 + x_6),$$
$$X_3 = x_1 x_4 x_6 + x_5^2 (x_2 + x_4),\quad X_4 = x_1 x_3 x_6 + x_5 x_6 (x_2 + x_4),$$
$$X_5 = x_1 x_2 x_6 + x_3 x_5 (x_2 + x_4),\quad X_6 = x_1^2 x_6 + x_4 x_5 (x_2 + x_4).$$

150. The homogeneous linear partial differential equations, which have as their system of integrals the quantities $f_1, \ldots, f_{2n-1}$ occurring in the normal form of Ω, have in § 148 been given with unknown forms occurring explicitly; but by the properties similar to that in § 145 the expressions can be changed so that only known forms shall occur.

The system of equations is

$$A_k f = \frac{\partial (f_1, \ldots, f_{2n-1}, f)}{\partial (x_1, \ldots, x_{2n-1}, x_{2n+k})} = 0,$$

which may be written

$$z_{2n} \frac{\partial f}{\partial x_{2n+k}} + \sum_{s=1}^{2n-1} z_{k,s} \frac{\partial f}{\partial x_s} = 0,$$

where

$$z_{2n} = \frac{\partial (f_1, \ldots, f_{2n-1})}{\partial (x_1, \ldots, x_{2n-1})}$$

and

$$z_{k,s} = \frac{\partial (f_1, \ldots, f_{2n-1})}{\partial_s (x_1, \ldots, x_{2n+k})},$$

the subscript in ∂_s implying that x_s is omitted from the sequence $x_1, \ldots, x_{2n-1}, x_{2n+k}$.

In § 145 it was proved that

$$\overset{n}{F_n} z_{2n} = \overset{n}{F_n} \frac{\partial (f_1, \ldots, f_{2n-1})}{\partial (x_1, \ldots, x_{2n-1})} = \sum_{r=1}^{2n-1} X_r [r+1, \ldots, r-1];$$

and similarly it may be proved that

$$\overset{n}{F_n} z_{k,s} = \overset{n}{F_n} \frac{\partial (f_1, \ldots, f_{2n-1})}{\partial_s (x_1, \ldots, x_{2n+k})} = \sum_{r=1}^{2n+k} X_r [r+1, \ldots, r-1],$$

where on the right-hand side the summation extends to the terms for which $r = 1, \ldots, s-1, s+1, \ldots, 2n-1, 2n+k$. Hence taking

$$y = z \overset{n}{F_n}$$

for each of the quantities z, the equations become

$$y_{2n}\frac{\partial f}{\partial x_{2n+k}}+\sum_{s=1}^{2n-1}y_{k,s}\frac{\partial f}{\partial x_s}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(A)},$$

for $k=0, 1, \ldots, q$*; and in this form of the equations all the coefficients of derivatives of f are expressed in terms of the quantities X which occur in Ω. In particular, that equation of the set (A), which corresponds to $k=0$, may be written

$$\sum_{s=1}^{2n} y_s\frac{\partial f}{\partial x_s}=0,$$

replacing $y_{0,s}$ by y_s; it is implicitly the same as the system of equations which occur in Pfaff's reduction and as the first equation in Clebsch's system†.

* These equations, except that for which $k=0$, are not given by Lie; each of them contains only $2n$ partial differential coefficients, instead of $2n+1$ such coefficients, as is the case with the succeeding equations and with Clebsch's system.

† The remaining equations in Clebsch's system may be obtained as follows. The equations

$$B_k f=\frac{\partial(f_1, \ldots, f_n, F_1, \ldots, F_n, f)}{\partial(x_1, \ldots\ldots\ldots\ldots\ldots\ldots, x_{2n}, x_{2n+k})}=0,$$

for $k=1, 2, \ldots, q$, are satisfied by $f=f_1, \ldots, f_n, F_1, \ldots, F_n$, that is, they are a system of q equations in $2n+q$ variables with $2n$ solutions independent of one another and they are therefore a complete system. Now in $B_k f$ the coefficient of $\dfrac{\partial f}{\partial x_s}$ is

$$\frac{\partial(f_1, \ldots\ldots\ldots, f_n, F_1, \ldots\ldots\ldots, F_n)}{\partial(x_1, \ldots, x_{s-1}, x_{s+1}, \ldots, x_{2n}, x_{2n+k})}$$
$$=[1, \ldots, s-1, s+1, \ldots, 2n, 2n+k],$$

a Pfaffian of order $2n$; and therefore the equation is

$$B_k f=\sum_{s=1}^{2n, 2n+k}\frac{\partial f}{\partial x_s}[1, \ldots, s-1, s+1, \ldots, 2n, 2n+k]=0,$$

each equation containing $2n+1$ derivatives of f.

These equations may be proved to be identical with Clebsch's system of similar equations, by means of properties of determinants. As in Clebsch's theory, the quantities $f_1, \ldots, f_{2n-1}$ are determined by the q equations

$$B_1 f=0,\ B_2 f=0, \ldots\ldots\ldots\ldots, B_q f=0$$

together with

$$\sum_{s=1}^{2n} y_s\frac{\partial f}{\partial x_s}=0,$$

which form also a complete system.

151. In order to complete the reduction of a differential expression which has a normal form of even character, it remains to construct the normal form of an unconditioned differential expression in an even number of variables.

Let such an expression be

$$\Omega_{2n} = X_1 dx_1 + X_2 dx_2 + \ldots\ldots + X_{2n} dx_{2n};$$

then the single partial differential equation determining the n functions, which give the differential elements of the normal form, and the $n-1$ independent ratios of the coefficients of those differential elements, is

$$y_1 \frac{\partial f}{\partial x_1} + y_2 \frac{\partial f}{\partial x_2} + \ldots \ldots + y_{2n} \frac{\partial f}{\partial x_{2n}} = 0,$$

where, as above,

$$(-1)^{r-1} y_r = \sum_{s=1}^{2n} X_s [s+1, \ldots\ldots, s-1],$$

and $s+1, \ldots, s-1$ are the integers $1, 2, \ldots, 2n$ in cyclical succession with s and r omitted and beginning with the integer next after s.

Let an integral of this differential equation be

$$g_1(x_1, \ldots, x_{2n}) = \text{constant} = a_1;$$

and suppose this equation solved so as to give x_{2n} explicitly in the form

$$x_{2n} = \gamma_1(x_1, \ldots, x_{2n-1}, a_1).$$

When substitution for x_{2n} takes place in Ω_{2n}, it becomes a differential expression in $2n-1$ variables, say Ω'_{2n-1}, the normal form of which contains $2n-2$ functions. If we have

$$\Omega'_{2n-1} = \Phi_1 d\phi_1 + \ldots\ldots + \Phi_{n-1} d\phi_{n-1},$$

then, replacing a_1 wherever it occurs by $g_1(x_1, \ldots, x_{2n})$, we have

$$\Omega_{2n} = \Phi_1 d\phi_1 + \ldots\ldots + \Phi_{n-1} d\phi_{n-1} + G_1 dg_1,$$

where G_1 is at once obtainable when the normal form of Ω'_{2n-1} is known.

Since the normal form of Ω'_{2n-1}, which involves $2n-1$ variables, contains only $2n-2$ functions, we apply a single substitution as in § 147 and change it into an unconditioned expression Ω_{2n-2}, from

the normal form of which that of Ω'_{2n-1} can be derived by the process of § 149.

The new expression Ω_{2n-2} is treated in the same manner as was Ω_{2n}; and so on in succession, until we come to an expression in two variables alone, which can be expressed in the form $G_n dg_n$.

Let the combination of the three operations which make the normal form of Ω_{2n} depend on that of Ω_{2n-2}, viz., (i) the derivation of some solution of the equation subsidiary to Ω_{2n}, (ii) a substitution in Ω_{2n} for one variable by means of that solution, (iii) a single Cauchy transformation applied to the expression modified by (ii); be termed a *reduction of order* $2n$.

152. Then the method of Lie for the reduction of a Pfaffian expression Ω in $2n+q$ variables, which has a normal form of even character in $2n$ functions, is generally as follows:—

The expression Ω is transformed to Ω_{2n} by q Cauchy transformations; Ω_{2n} is made to depend upon Ω_{2n-2} by a reduction of order $2n$; Ω_{2n-2} upon Ω_{2n-4} by a reduction of order $2n-2$; and so on in succession, until an expression Ω_2 is obtained. When this is integrated into a single term, then definite and explicit operations lead to the normal forms of $\Omega_4, \Omega_6, \ldots, \Omega_{2n}, \Omega_{2n+q}$.

153. We now pass to the discussion of the supposition which is the alternative of that adopted in § 147, viz., that a differential expression

$$\Omega = \sum_{i=1}^{2n+q} X_i dx_i$$

is recognised, by the tests of § 144, to have a normal form of uneven character in $2n+1$ functions, say

$$df_0 + \sum_{i=1}^{n} F_i df_i.$$

The method is the same as that already adopted in §§ 147—152. As the analytical details of the proofs of the various results are very similar to those in the investigation just completed, it will be sufficient merely to state these results.

154. It is assumed (§ 145) that

$$\sum_{s=1}^{2n+1} X_s [s+1, \ldots, s-1]$$

does not vanish and that the coefficient of X_{2n+1} in this expression, the Pfaffian $[1, \ldots, 2n]$, does not vanish.

(i) The substitutions

$$x_{2n+k} = a_{2n+k} + (x_{2n+1} - a_{2n+1})\, y_k,$$

for $k = 2, \ldots, q$, when applied to Ω, transform it into an expression Ω'_{2n+1} in $2n+1$ variables, the normal form of which contains $2n+1$ functions.

(ii) Let a normal form of Ω'_{2n+1} be

$$d\phi_0 + \sum_{i=1}^{n} \Phi_i d\phi_i,$$

where the quantities ϕ and Φ are functions of the variables $x_1, \ldots, x_{2n+1}$ and of the quantities y and a; and solve the $2n$ equations

$$\phi_i(x_1, \ldots, x_{2n}, x_{2n+1}, y_2, \ldots, y_q) = \phi_i(h_1, \ldots, h_{2n}, a_{2n+1}, y_2, \ldots, y_q),$$

$$\Phi_i(x_1, \ldots, x_{2n}, x_{2n+1}, y_2, \ldots, y_q) = \Phi_i(h_1, \ldots, h_{2n}, a_{2n+1}, y_2, \ldots, y_q),$$

for the $2n$ quantities $h_1, \ldots, h_{2n}$ as functions of the variables. After their values have been found, replace the y's by their values in terms of the variables x: suppose that the equations are

$$h_\mu = h_\mu(x_1, \ldots, x_{2n+q})$$

for $\mu = 1, 2, \ldots, 2n$. Then a normal form of the original expression is

$$d\phi_0 + \sum_{i=1}^{n} \Phi_i(h_1, \ldots, h_{2n}, a_{2n+1}, y_2, \ldots, y_q)\, d\phi_i(h_1, \ldots, h_{2n}, a_{2n+1}, y_2, \ldots, y_q),$$

where ϕ_0 is determined by a quadrature

$$\phi_0 = \int \left\{ \sum_{i=1}^{2n+q} X_i dx_i - \sum_{i=1}^{n} \Phi_i(h, y,)\, d\phi_i(h, y) \right\}.$$

(iii) Let $\theta_{k,s}$ denote the Pfaffian $[s+1, \ldots, s-1]$, where $s+1, \ldots, s-1$ are the integers $1, 2, \ldots, 2n, 2n+k$ in cyclical order with the omission of s and begin with the integer next after s in the succession. Then the system of q equations which have $f_1, \ldots, f_n, F_1, \ldots, F_n$ as their integral system is

$$A_k f = \sum_{s=1}^{2n} \theta_{k,s} \frac{\partial f}{\partial x_s} + [1, \ldots, 2n] \frac{\partial f}{\partial x_{2n+k}} = 0.$$

(iv) To obtain the normal form of an unconditioned expression Ω_{2n+1} in $2n+1$ variables, the process is a series of reductions as in § 152.

Let any integral

$$g_1(x_1, \ldots, x_{2n+1}) = a_1$$

of the single subsidiary equation

$$\sum_{i=1}^{2n+1} \frac{\partial f}{\partial x_i} [i+1, \ldots\ldots, i-1] = 0$$

be obtained, and be solved for x_{2n+1} in the form

$$x_{2n+1} = \gamma_1(x_1, \ldots, x_{2n}, a_1).$$

When substitution is made for x_{2n+1} in Ω_{2n+1} by this equation, the new expression involves $2n$ variables and has $2n-1$ functions in its normal form. Applying to this new expression a single Cauchy substitution as above, we obtain an expression Ω_{2n-1} in $2n-1$ variables with a normal form which has $2n-1$ functions, i.e., it is unconditioned. This group of operations enabling us to pass from Ω_{2n+1} to Ω_{2n-1} may be called a *reduction of order* $2n+1$.

(v) A reduction of order $2n+1$ is applied to Ω'_{2n+1} of (i) and leads to Ω'_{2n-1}; a reduction of order $2n-1$ is applied to Ω'_{2n-1} and leads to Ω'_{2n-3}; and so on in succession, until we come to Ω'_1 which is a perfect differential. Then definite operations lead to the normal forms of Ω'_3, $\Omega'_5, \ldots$, Ω'_{2n+1} and thence, by (ii), to the normal form of Ω_{2n+q}.

Ex. There is an immediate corollary from Lie's method for a Pfaffian expression, in the form of a process for integrating an exact equation.

Let $\sum_{i=1}^{m} X_i dx_i = 0$ be an exact equation, so that the differential expression on the left-hand side can be put into the form $\Theta d\theta$. Applying the substitutions

$$x_r = a_r + (x_2 - a_2) y_r$$

for $r = 3, \ldots\ldots, m$, the expression takes the form

$$Y_1 dx_1 + Y_2 dx_2.$$

Let

$$Y_1 dx_1 + Y_2 dx_2 = \Phi d\phi,$$

where ϕ and Φ are functions of x_1, x_2 and of the quantities a and y. Then solving the equation

$$\phi(x_1, x_2, y_3, \ldots\ldots, y_m) = \phi(a_1, h, y_3, \ldots\ldots, y_m)$$

for h and eliminating the quantities y from the expression, it follows at once from the general theory that

$$\sum_{i=1}^{m} X_i dx_i = H dh,$$

where H is some function, which can be determined at once from any one of the equations

$$X_s = H \frac{\partial h}{\partial x_s};$$

and thus an integral of the equation is given by

$$h = \text{constant},$$

to which all integrals (§ 3) are equivalent.

CHAPTER XI.

FROBENIUS' METHOD.

THE investigations of Frobenius in the formulation and solution of Pfaff's problem * deal rather with the general theory of the reduction of the expression to a normal form than with any processes for the integration of equations which occur in the reduction; and the interest lies chiefly in the purely algebraical association of the number of terms in the reduced form with the critical conditions—the same in number and form as Natani's—satisfied by the coefficients of the original expression.

155. Let the expression $\sum_{i=1}^{n} X_i dx_i$ become $\sum_{i=1}^{n} X_i' dx_i'$ by means of transforming relations

$$x_i = x_i(x_1', \ldots, x_n'),$$

so that, if x_{ij} denote $\partial x_i/\partial x_j'$, the differential elements are connected by the linear relations

$$dx_\alpha = \sum_{i=1}^{n} x_{\alpha i}\, dx_i' \qquad (\alpha = 1, \ldots, n).$$

In the equation

$$\sum_{i=1}^{n} X_i dx_i = \sum_{i=1}^{n} X_i' dx_i',$$

the system of variations dx (and therefore also the system dx') are arbitrary and independent so far as the variations are concerned; so that, if δx (with $\delta x'$ in consequence) be other variations, we have also

$$\sum_{i=1}^{n} X_i\, \delta x_i = \sum_{i=1}^{n} X_i' \delta x_i'.$$

* The chief part of his exposition is contained in his memoir "Ueber das Pfaff'sche Problem," *Crelle*, t. lxxxii. (1877), pp. 230—315; other amplifications were given by him in another memoir "Ueber homogene totale Differentialgleichungen," *Crelle*, t. lxxxvi. (1879), pp. 1—19.

Now taking variations of each of these pairs of equal quantities in the forms

$$\delta \sum_{i=1}^{n} (X_i dx_i) = \delta \sum_{i=1}^{n} (X_i' dx_i'),$$

$$d \sum_{i=1}^{n} (X_i \delta x_i) = d \sum_{i=1}^{n} (X_i' \delta x_i'),$$

we have

$$\delta \sum_{i=1}^{n} X_i dx_i - d \sum_{i=1}^{n} X_i \delta x_i = \delta \sum_{i=1}^{n} X_i' dx_i' - d \sum_{i=1}^{n} X_i' \delta x_i',$$

or, since

$$d\delta x = \delta dx$$

because the variations are arbitrary, it follows that

$$\sum_{i=1}^{n} (\delta X_i dx_i - dX_i \delta x_i) = \sum_{i=1}^{n} (\delta X_i' dx_i' - dX_i' \delta x_i'),$$

and therefore

$$\sum_{i,j} \left(\frac{\partial X_i}{\partial x_j} dx_i \delta x_j - \frac{\partial X_i}{\partial x_j} dx_j \delta x_i\right) = \sum_{i,j} \left(\frac{\partial X_i'}{\partial x_j'} \delta x_j' \, dx_i' - \frac{\partial X_i'}{\partial x_j'} dx_j' \, \delta x_i'\right),$$

so that

$$\sum_{i,j} a_{ij} \, dx_i \, \delta x_j = \sum_{i,j} a_{ij}' \, dx_i' \, \delta x_j',$$

an equation which is a necessary consequence of the original variational equations. Thus the expression

$$\sum_{i,j} a_{ij} \, dx_i \, \delta x_j$$

is a bilinear covariant, associated with the original differential expression.

If we take a third variation Δx, different from and independent of the two already adopted, and proceed as above from the equations

$$\Delta \Sigma a_{ij} \, dx_i \, \delta x_j = \Delta \Sigma a_{ij}' \, dx_i' \, \delta x_j',$$

$$\delta \Sigma a_{ij} \, \Delta x_i \, dx_j = \delta \Sigma a_{ij}' \, \Delta x_i' \, dx_j',$$

$$d \Sigma a_{ij} \, \delta x_i \, \Delta x_j = d \Sigma a_{ij}' \, \delta x_i' \, \Delta x_j',$$

combining them so as to have a trilinear covariant, the coefficient of $dx_i \, \delta x_j \, \Delta x_k$ is

$$\frac{\partial a_{ij}}{\partial x_k} + \frac{\partial a_{jk}}{\partial x_i} + \frac{\partial a_{ki}}{\partial x_j}$$

and therefore is zero. Hence for the Pfaffian expression the trilinear

covariant is evanescent; and so for all successive multilinear covariants*; and therefore we need only consider the linear expression

$$\Sigma X dx$$

(which implies also $\Sigma X \delta x$) and the bilinear expression

$$\Sigma a_{ij}\, dx_i\, \delta x_j,$$

which are covariantive functions of index zero.

The original system of linear and bilinear expressions is said to be *equivalent* to the transformed system.

156. Now if we replace dx_i by u_i and δx_i by v_i, so that the u's and v's are two sets of variables which are independent of one another and are transformed by the same substitutions

$$u_a = \sum_{i=1}^{n} x_{ai}\, u_i',$$

then the original forms are

$$\sum_{a=1}^{n} X_a u_a, \quad \sum_{i,j} a_{ij} u_i v_j.$$

When these are subjected to the foregoing linear transformations, they take the similar and equivalent forms

$$\sum_{a=1}^{n} X_a' u_a', \quad \sum_{i,j} a_{ij}' u_i' v_j',$$

where

$$X_a' = \sum_{i=1}^{n} x_{ia} X_i,$$

so as to make

$$a_{ij}' = \sum_{a,\beta}^{n} x_{ai}\, x_{\beta j}\, a_{a\beta};$$

so that we have a merely algebraical transformation between equivalent systems of two simultaneous forms.

If now we take the converse question and assume that two simultaneous forms

$$\sum_{a=1}^{n} X_a' u_a', \qquad \sum_{i,j} a_{ij}' u_i' v_j',$$

are equivalent to the two forms

$$\sum_{a=1}^{n} X_a u_a, \qquad \sum_{i,j} a_{ij} u_i v_j,$$

* For bilinear expressions and associated covariants, not connected with linear Pfaffians, see Christoffel, *Crelle*, t. lxx. (1869), pp. 46—70; Lipschitz, *ib.*, pp. 71—102.

then it is first necessary to find all the algebraical transformations which render this equivalence possible. It does not however follow that such an algebraical transformation leads to a differential transformation which will give the equation

$$\sum_{i=1}^{n} X_i\, dx_i = \sum_{i=1}^{n} X_i'\, dx_i' ;$$

the algebraical transformation is useful for our purpose only if the inferred differential transformations leave the expressions for the elements dx perfect differentials. Such as do this necessarily reproduce the bilinear differential covariant.

It thus follows that, in order to effect the transformation of a Pfaffian expression by this method, there are two parts in the investigation. One, purely algebraical, is the derivation of all the substitutions which will change a system of two forms

$$\sum_{i=1}^{n} X_i u_i, \quad \sum_{i,j} a_{ij} u_i v_j,$$

into an equivalent system; the other is an examination of the analytical capability of such transformations when they are changed into differential substitutions.

157. When the linear substitutions are made in the bilinear form

$$W = \sum_{i,j}^{n} a_{ij} u_i v_j,$$

so as to transform it to

$$W' = \sum_{i,j}^{n} a_{ij}' u_i' v_j',$$

then the coefficients in the two forms are connected by the relations

$$a_{ij}' = \sum_{\alpha,\beta}^{n} x_{\alpha i}\, x_{\beta j}\, a_{\alpha\beta} ;$$

and any determinant* of the m^{th} order in the coefficients a_{ij}' is a homogeneous linear function of determinants of the m^{th} order in the coefficients a_{ij}.

The reciprocal of this relation occurs when the inverse substitutions are applied to W'.

* Scott's *Determinants*, p. 53.

Hence the determinants of any one order in the coefficients a_{ij}' and those of the same order in the coefficients a_{ij} vanish together, and therefore the highest order of non-vanishing determinants of coefficients is the same for the transformed as for the original bilinear expression, that is, the highest order of non-vanishing determinants of the coefficients is an invariant for the linear substitutions.

In order to consider the simultaneous transformation of the forms

$$\sum_{i,j}^{n} a_{ij} u_i v_j, \quad \sum_{i=1}^{n} X_i u_i, \quad \sum_{i=1}^{n} X_i v_i,$$

it is convenient to construct a new bilinear form

$$\Theta = \sum_{i,j}^{n} a_{ij} u_i v_j + v_{n+1} \sum_{i=1}^{n} X_i u_i + u_{n+1} \sum_{i=1}^{n} X_i v_i + A u_{n+1} v_{n+1},$$

where A denotes an arbitrary unchanging magnitude and the new variables are subject to the transformations $u_{n+1} = u'_{n+1}$, $v_{n+1} = v'_{n+1}$. Then the order in the coefficients of Θ of the highest non-vanishing determinants is an invariant for the linear substitutions.

158. Now all these determinants are minors of the complete determinant of order n (in the second case of order $n+1$) involving all the coefficients; and therefore there are, in the present case, two invariantive integers. The first, say m, is the order of the highest non-vanishing minor in the skew determinant

$$\Delta_1 = \begin{vmatrix} a_{11}, & a_{12}, & \dots\dots, & a_{1n} \\ a_{21}, & a_{22}, & \dots\dots, & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1}, & a_{n2}, & \dots\dots, & a_{nn} \end{vmatrix}.$$

The second, say m', is the order of the highest non-vanishing minor in the determinant

$$\begin{vmatrix} a_{11}, & a_{12}, & \dots\dots, & a_{1n}, & X_1 \\ a_{21}, & a_{22}, & \dots\dots, & a_{2n}, & X_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1}, & a_{n2}, & \dots\dots, & a_{nn}, & X_n \\ X_1, & X_2, & \dots\dots, & X_n, & A \end{vmatrix},$$

where A is arbitrary. Now this order does not depend upon the value of the arbitrary quantity and must be the same whatever

value be assigned to A. Taking then zero as the value and changing the signs of all the terms in the last line so as to make the determinant explicitly skew, we have the second invariantive integer m' as the order of the highest non-vanishing minor in the skew determinant

$$\Delta_2 = \begin{vmatrix} a_{11}, & a_{12}, & \ldots\ldots, & a_{1n}, & X_1 \\ a_{21}, & a_{22}, & \ldots\ldots, & a_{2n}, & X_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1}, & a_{n2}, & \ldots\ldots, & a_{nn}, & X_n \\ -X_1, & -X_2, & \ldots\ldots, & -X_n, & 0 \end{vmatrix}.$$

The integers m and m' must be even. For, among the non-vanishing minors of these orders there must be principal minors, which are skew determinants since Δ_1 and Δ_2 are skew; let such an one for Δ_1 be

$$\begin{vmatrix} a_{11}, & a_{12}, & \ldots\ldots, & a_{1m} \\ a_{21}, & a_{22}, & \ldots\ldots, & a_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1}, & a_{m2}, & \ldots\ldots, & a_{mm} \end{vmatrix}.$$

Then its order must be even; and hence we take m, the invariantive integer associated with Δ_1, to be $2r$. Similarly for m', the invariantive integer associated with Δ_2. Now m' cannot be less than $2r$; and it cannot be greater than $2r+2$, for all determinants of order greater than $m+2$ would then have non-vanishing minors of order greater than m which would at the same time be minors of Δ_1.

If all the minors in Δ_2 of order $2r+2$ vanish, then all those of order $2r+1$ vanish; so that, if m' be not $2r+2$, it must be $2r$, for by the hypothesis as to Δ_1 the minors of Δ_2 of order $2r$ cannot all vanish.

There are thus two cases which arise:—

(i) the invariantive integer of Δ_1 being $2r$, that of Δ_2 is $2r+2$;

(ii) the invariantive integer of Δ_1 being $2r$, that of Δ_2 is $2r$.

If instead of considering the two invariantive integers of either of the cases, we consider their arithmetic mean, this mean is sufficient to determine the two integers. For, if it be an even number, the invariantive integers are equal and so the second case occurs; and, if it be an uneven number, the invariantive integers are unequal and so the first case occurs. Hence it is sufficient to

consider only *a single invariant p, the arithmetic mean of the invariantive integers* associated with Δ_1 and Δ_2 respectively*.

The *invariance of the integer p is a necessary inference from the equivalence of two systems of forms; we now proceed to shew that it is a sufficient condition for their equivalence*, by obtaining (on the assumption of the invariance) equations of transformation.

159. Let $z_1, z_2, \ldots, z_k$ be k independent functions of $x_1, \ldots, x_n$ which by the (at present unknown) transformations become $z_1', z_2', \ldots, z_k'$; then the expression

$$x_0 \sum_{i=1}^{n} X_i dx_i + \sum_{i=1}^{k} x_{n+i}\, dz_i \text{ or, say, } \sum_{i=0}^{n+k} Y_i dx_i$$

changes, by those transformations together with

$$x_0' = x_0, \quad x'_{n+i} = x_{n+i},$$

into

$$x_0' \sum_{i=1}^{n} X_i' dx_i' + \sum_{i=1}^{k} x'_{n+i}\, dz_i' \text{ or, say, into } \sum_{i=0}^{n+k} Y_i' dx_i'.$$

Hence there is an invariantive integer, being the order of the highest non-vanishing minors of the determinant, which is associated with the expression $\Sigma Y dx$.

Now

$$Y_0 = 0, \quad Y_{n+i} = 0 \quad (i = 1, \ldots, k);$$

$$Y_r = x_0 X_r + \sum_{i=1}^{k} x_{n+i} \frac{\partial z_i}{\partial x_r} \quad (r = 1, \ldots, n);$$

and therefore, when we take the elements of the associated determinant in the form

$$\frac{\partial Y_s}{\partial x_t} - \frac{\partial Y_t}{\partial x_s}$$

* It is easy to infer from Natani's conditions (§§ 99, 100) that, if p be even, there is an even number p of independent functions in the normal form of a Pfaffian expression and that, if p be odd, there is an odd number p of such functions in the normal form. The combination of this method of statement of Natani's conditions with the invariantive character of p inferred by Frobenius leads to some of Lie's results (§ 142) relative to the persistence of character of a normal form.

for $s, t = 0, 1, \ldots, n+k$, the determinant is

$$\Delta_3' = \begin{vmatrix} x_0 a_{11}, & \ldots\ldots, & x_0 a_{1n}, & X_1, & \dfrac{\partial z_1}{\partial x_1}, & \ldots\ldots, & \dfrac{\partial z_k}{\partial x_1} \\ x_0 a_{21}, & \ldots\ldots, & x_0 a_{2n}, & X_2, & \dfrac{\partial z_1}{\partial x_2}, & \ldots\ldots, & \dfrac{\partial z_k}{\partial x_2} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ x_0 a_{n1}, & \ldots\ldots, & x_0 a_{nn}, & X_n, & \dfrac{\partial z_1}{\partial x_n}, & \ldots\ldots, & \dfrac{\partial z_k}{\partial x_n} \\ -X_1, & \ldots\ldots, & -X_n, & 0, & 0, & \ldots\ldots, & 0 \\ -\dfrac{\partial z_1}{\partial x_1}, & \ldots\ldots, & -\dfrac{\partial z_1}{\partial x_n}, & 0, & 0, & \ldots\ldots, & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ -\dfrac{\partial z_k}{\partial x_1}, & \ldots\ldots, & -\dfrac{\partial z_k}{\partial x_n}, & 0, & 0, & \ldots\ldots, & 0 \end{vmatrix}.$$

Non-evanescent determinants which are minors of Δ_3' are of order less than n; and all such contain some power of x_0 as a factor and contain x_0 only in that form. Hence, so far as the order of vanishing minors is concerned, we may take $x_0 = 1$; and the corresponding form of Δ_3' will be denoted by Δ_3, so that

$$\Delta_3 = \Delta_3', \text{ when } x_0 = 1.$$

Let the invariantive integer of Δ_2 be m $(= 2r)$, so that the invariantive integer of Δ_1 is either $2r$ or $2r-2$. Then the invariantive integer of Δ_3 is in general greater than $2r$, so long as the functions z are perfectly arbitrary; but it can be diminished by determining the quantities z, so that they satisfy the partial differential equations which are the conditions for the diminution of the highest order of non-vanishing minors. It cannot, however, be reduced below $2r$, on account of the hypothesis which has been made about Δ_2; and therefore we shall suppose that the invariantive integer of Δ_3 is $2r$, the same as that of Δ_2. Thus $z_1, \ldots, z_k$ will satisfy a number of partial differential equations; and we proceed to determine the greatest possible number of functions z, which will make the invariantive integer of the associated determinant Δ_3 the same as that of Δ_2, viz., $2r$.

160. Suppose, then, that k functions z independent of one another have been obtained (k may be zero), such that all minors in Δ_3 of order $m+1$ vanish but not all those of lower order, so that m is the invariantive integer of Δ_3. If a determinant,

similar to Δ_2 but containing $k+1$ functions z, be constructed in the form

$$\left|\begin{array}{cccccccc}
a_{11}, & \ldots\ldots, & a_{1n}, & X_1, & \frac{\partial z_1}{\partial x_1}, & \ldots\ldots, & \frac{\partial z_{k+1}}{\partial x_1} \\
\multicolumn{7}{c}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots} \\
a_{n1}, & \ldots\ldots, & a_{nn}, & X_n, & \frac{\partial z_1}{\partial x_n}, & \ldots\ldots, & \frac{\partial z_{k+1}}{\partial x_n} \\
-X_1, & \ldots\ldots, & -X_n, & 0, & 0, & \ldots\ldots, & 0 \\
-\frac{\partial z_1}{\partial x_1}, & \ldots\ldots, & -\frac{\partial z_1}{\partial x_n}, & 0, & 0, & \ldots\ldots, & 0 \\
\multicolumn{7}{c}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots} \\
-\frac{\partial z_{k+1}}{\partial x_1}, & \ldots\ldots, & -\frac{\partial z_{k+1}}{\partial x_n}, & 0, & 0, & \ldots\ldots, & 0
\end{array}\right|,$$

then, in order that its invariantive integer may be $2r$, the same as that of Δ_2, it is necessary and sufficient that z_{k+1} should satisfy the linear equations

$$u_1\frac{\partial f}{\partial x_1}+u_2\frac{\partial f}{\partial x_2}+\ldots\ldots+u_n\frac{\partial f}{\partial x_n}=0 \quad \ldots\ldots\ldots\ldots(1)$$

for all values of the quantities u which allow the relations

$$\left.\begin{array}{lll}
\sum_{i=1}^{n} a_{ri}u_i + X_r u + \sum_{j=1}^{k}\frac{\partial z_j}{\partial x_r}u_{n+j} = 0 & & (r=1,\ldots,n) \\
\sum_{i=1}^{n} X_i u_i & = 0 & \\
\sum_{i=1}^{n} \frac{\partial z_j}{\partial x_i}u_i & = 0 & (j=1,\ldots,k)
\end{array}\right\}\ldots(2)$$

to be satisfied. But, on account of the fact that minors of Δ_2 or Δ_2' of order higher than m vanish and the consequent non-independence of equations in (2), it follows that all values of the quantities u, which allow the relations (2) to be satisfied, allow also the relations

$$\left.\begin{array}{lll}
\sum_{i=1}^{n} x_0 a_{ri}u_i + X_r u + \sum_{j=1}^{k}\frac{\partial z_j}{\partial x_r}u_{n+j} = 0 & & (r=1,\ldots,n) \\
\sum_{i=1}^{n} X_i u_i & = 0 & \\
\sum_{i=1}^{n} \frac{\partial z_j}{\partial x_i}u_i & = 0 & (j=1,\ldots,k)
\end{array}\right\}\ldots(2)'$$

to be satisfied. And now the new quantity z_{k+1} satisfies the system of differential equations (1) for all values of the quantities u limited by the relations (2)$'$.

161. Now when we bear in mind the invariantive character of the determinants Δ_1, Δ_2, Δ_3, and the relations of the linear and the bipartite forms to the Pfaffian and its bilinear covariant and when we notice that these determinants are eliminants of quantities of the form $\frac{\partial W}{\partial u_i}$ $\left(\text{or of quantities of the form } \frac{\partial W}{\partial v_j}\right)$, where W is a bipartite form, we see that the conditions of § 156 can be satisfied as follows. *It is necessary that merely algebraical transformations shall*, when modified into substitutions for the original Pfaffian, furnish elements dx which, by being complete differentials, will *lead to integral equations of substitution for the Pfaffian*; and this will be effected if, in the equations (1) and the relations (2)′ we actually replace the variables u by differential elements dx, these elements being taken to be complete differentials.

When this modification is made, the result of § 160 is that z_{k+1} satisfies the equation

$$\frac{\partial f}{\partial x_1} dx_1 + \ldots\ldots + \frac{\partial f}{\partial x_n} dx_n = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(I)}$$

for all variations dx which are consistent with the equations

$$\left.\begin{array}{ll} \sum_{i=1}^{n} x_0 a_{ri} dx_i + X_r dx_0 + \sum_{j=1}^{k} \frac{\partial z_j}{\partial x_r} dx_{n+j} = 0 & (r = 1, \ldots, n) \\ \sum_{i=1}^{n} X_i dx_i = 0 & \\ \sum_{i=1}^{n} \frac{\partial z_j}{\partial x_i} dx_i = 0 & (j = 1, \ldots, k) \end{array}\right\} \text{(II)}.$$

The initial hypothesis implied that all the functions z were to be functions of $x_1, \ldots, x_n$ alone; and therefore equation (I) takes the form

$$dz_{k+1} = 0,$$

or z_{k+1} *is an integral of the system of equations* (II). Also it is manifest, from the same reason, that $z_1, \ldots, z_k$, derived from the last k equations of (II), are also integrals; so that, regarding (II) as a system of equations to be integrated, we have the functions z as integrals of this system.

The coefficients of the differential elements on the left-hand sides of the equations (II) are the various quantities

$$\frac{\partial Y_r}{\partial x_s} - \frac{\partial Y_s}{\partial x_r},$$

where Y_r and Y_s (for $r, s = 0, 1, \ldots, n+k$) are any two of the coefficients of a Pfaffian expression $\sum_{i=0}^{n+k} Y_i dx_i$; and therefore (*Ex.*, § 31) the system is complete. Moreover, since the minors of order m in the determinant of the left-hand sides do not all vanish while all minors of order higher than m do vanish, there are m equations (and not more than m equations) independent of one another; and therefore the system is equivalent to an exact system of m equations. Hence the system (II) has m independent integrals.

The functions z are to involve only the variables $x_1, \ldots, x_n$; the system (II) involves not alone these variables but also $x_0, x_{n+1}, \ldots, x_{n+k}$; the integrals of the system must therefore include the whole aggregate of variables. The m independent integrals can be replaced by m independent functional combinations of them; and if we choose such functional combinations as to include the greatest possible number of equations which are free from the variables $x_0, x_{n+1}, \ldots, x_{n+k}$, we shall have $m-k-1$ equations independent of one another and involving only the variables $x_1, \ldots, x_n$ and constants, and $k+1$ equations independent of one another and involving the variables $x_0, x_{n+1}, \ldots, x_{n+k}$. Now each of these leads to an integral of (II), that is, to a function z such that

$$dz = 0$$

in virtue of the equations (II) and the system of integrals of these equations; there are therefore $m-k-1$ independent integrals of the system (II) which involve only the variables $x_1, \ldots, x_n$ and are therefore of the type of functions z.

But k such functions z have been supposed to be obtained, the supposition being placed in evidence by the last k equations of (II); hence there are $(m-k-1)-k$, that is, $m-2k-1$ further integrals of the required type. When m is $2r$ and k is $r-1$ (so that $r-1$ integrals are supposed known), the value of $m-2k-1$ is unity, and therefore one more integral can be found. Hence it follows that *r independent functions $z_1, \ldots, z_r$ can be found such that all minors of order greater than $2r$ in*

$$\begin{vmatrix} a_{11} & ,\ldots\ldots, & a_{1n} & , X_1, & \dfrac{\partial z_1}{\partial x_1} & ,\ldots\ldots, & \dfrac{\partial z_r}{\partial x_1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & ,\ldots\ldots, & a_{nn} & , X_n, & \dfrac{\partial z_1}{\partial x_n} & ,\ldots\ldots, & \dfrac{\partial z_r}{\partial x_n} \\ -X_1 & ,\ldots\ldots, & -X_n, & 0, & 0 & ,\ldots\ldots, & 0 \\ -\dfrac{\partial z_1}{\partial x_1} & ,\ldots\ldots, & -\dfrac{\partial z_1}{\partial x_n}, & 0, & 0 & ,\ldots\ldots, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\dfrac{\partial z_r}{\partial x_1} & ,\ldots\ldots, & -\dfrac{\partial z_r}{\partial x_n}, & 0, & 0 & ,\ldots\ldots, & 0 \end{vmatrix}$$

vanish.

162. Since all minors of order higher than $2r$ vanish, it follows that, among others, all principal minors of order $2r+2$ vanish. If then $\alpha, \ldots, \epsilon$ be any selection of $r+1$ integers from the series $1, 2, \ldots, n$, such a minor is

$$\begin{vmatrix} a_{\alpha\alpha} & ,\ldots\ldots, & a_{\alpha\epsilon} & , X_\alpha, & \dfrac{\delta z_1}{\partial x_\alpha} & ,\ldots\ldots, & \dfrac{\partial z_r}{\partial x_\alpha} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{\epsilon\alpha} & ,\ldots\ldots, & a_{\epsilon\epsilon} & , X_\epsilon, & \dfrac{\partial z_1}{\partial x_\epsilon} & ,\ldots\ldots, & \dfrac{\partial z_r}{\partial x_\epsilon} \\ -X_\alpha & ,\ldots\ldots, & -X_\epsilon, & 0, & 0 & ,\ldots\ldots, & 0 \\ -\dfrac{\partial z_1}{\partial x_\alpha} & ,\ldots\ldots, & -\dfrac{\partial z_1}{\delta x_\epsilon}, & 0, & 0 & ,\ldots\ldots, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\dfrac{\partial z_r}{\partial x_\alpha} & ,\ldots\ldots, & -\dfrac{\partial z_r}{\partial x_\epsilon}, & 0, & 0 & ,\ldots\ldots, & 0 \end{vmatrix},$$

the value of which is the square of

$$\begin{vmatrix} X_\alpha & ,\ldots\ldots\ldots, & X_\epsilon \\ \dfrac{\partial z_1}{\partial x_\alpha} & ,\ldots\ldots\ldots, & \dfrac{\partial z_1}{\partial x_\epsilon} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial z_r}{\partial x_\alpha} & ,\ldots\ldots\ldots, & \dfrac{\partial z_r}{\partial x_\epsilon} \end{vmatrix};$$

and therefore all such minors as may arise from selections of $r+1$ integers from the series $1, \ldots, n$ vanish, that is, all the determinants of the $(r+1)^{\text{th}}$ degree of the system

$$\ldots\left\|\begin{matrix} X_1, & X_2, & \ldots\ldots\ldots, & X_n \\ \dfrac{\partial z_1}{\partial x_1}, & \dfrac{\partial z_1}{\partial x_2}, & \ldots\ldots\ldots, & \dfrac{\partial z_1}{\partial x_n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \dfrac{\partial z_r}{\partial x_1}, & \dfrac{\partial z_r}{\partial x_2}, & \ldots\ldots\ldots, & \dfrac{\partial z_r}{\partial x_n} \end{matrix}\right\|\ldots$$

vanish. Hence r quantities, say $Z_1, \ldots, Z_r$, can be determined such that the n equations

$$X_s = \sum_{i=1}^{r} Z_i \frac{\partial z_i}{\partial x_s}$$

for $s = 1, 2, \ldots, n$ are satisfied: and from these we have

$$\sum_{s=1}^{n} X_s dx_s = \sum_{s=1}^{n} \sum_{i=1}^{r} Z_i \frac{\partial z_i}{\partial x_s} dx_s,$$

$$= \sum_{i=1}^{r} Z_i dz_i,$$

because z_i is a function of $x_1, \ldots, x_n$ only and the quantities dx are perfect differentials.

Further, since all principal minors of order less than $2r+2$ do not vanish, we cannot establish a system of relations of the form

$$X_s = \sum_{i=1} Z_i \frac{\partial z_i}{\partial x_s}$$

involving a number of functions z less than r; and therefore we cannot infer a transformation of $\sum_{s=1}^{n} X_s dx_s$, which shall involve a number of differential elements dz less than r.

163. Hitherto we have used the single assumption (§ 159) that the invariantive integer of Δ_2 is $2r$; we now must consider the invariantive integer of Δ_1, which may be either $2r$ or $2r-2$. The two cases will be taken in turn.

164. (i) First, let the invariantive integer of Δ_1 be $2r$; so that the minors of order $2r$ in Δ_1 do not all vanish. Now

$$\begin{aligned} a_{ij} &= \frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i} \\ &= \sum_{s=1}^{r} \left(\frac{\partial Z_s}{\partial x_j} \frac{\partial z_s}{\partial x_i} - \frac{\partial Z_s}{\partial x_i} \frac{\partial z_s}{\partial x_j} \right) \end{aligned}$$

from the above values obtained for X_s; and therefore the minors of Δ_1 of order $2r$ are obtained by multiplying some one determinant of order $2r$ of the system

$$\cdots \left\| \frac{\partial z_1}{\partial x_s}, \ldots\ldots, \frac{\partial z_r}{\partial x_s}, \frac{\partial Z_1}{\partial x_s}, \ldots\ldots, \frac{\partial Z_r}{\partial x_s} \right\| \cdots$$

by a determinant of order $2r$ of the system

$$\cdots \left\| \frac{\partial Z_1}{\partial x_t}, \ldots\ldots, \frac{\partial Z_r}{\partial x_t}, -\frac{\partial z_1}{\partial x_t}, \ldots\ldots, -\frac{\partial z_r}{\partial x_t} \right\| \cdots$$

for $s, t = 1, 2, \ldots, n$. Since then the products do not all vanish, the determinants of order $2r$ of these systems do not all vanish; and therefore the quantities $z_1, \ldots, z_r, Z_1, \ldots, Z_r$ are independent of one another.

By the invariance of the two integers, each $2r$, associated with Δ_1 and Δ_2, it follows that an expression $\sum_{s=1}^{n} X_s' dx_s'$, which is derived from the expression $\sum_{i=1}^{n} X_s dx_s$ by the (unknown) equations of transformation, can be expressed in the form

$$\sum_{i=1}^{r} Z_i' dz_i',$$

where the $2r$ quantities $z_1', \ldots, z_r', Z_1', \ldots, Z_r'$ are independent of one another; and this result, it may be repeated, is a consequence of the simultaneous invariance of the two integers. Now, as explained in § 158, these two integers may be replaced by the single invariantive integer p, their arithmetic mean, which in the present case is equal to $2r$.

Since the $2r$ quantities z and Z are independent of one another, as are also the $2r$ quantities z' and Z', the simplest equations of relation which will transform $\sum_{i=1}^{r} Z_i dz_i$ into $\sum_{i=1}^{r} Z_i' dz_i'$ are

$$z_i = z_i', \quad Z_i = Z_i', \qquad (i = 1, 2, \ldots, r).$$

Substituting for z, Z, z', Z' their respective values in terms of x and of x', we have equations of relation which are *sufficient* to give the equation

$$\sum_{s=1}^{n} X_s dx_s = \sum_{s=1}^{n} X_s' dx_s';$$

and these equations have been obtained on the assumption of the *invariantive persistence of the even integer* $p = 2r$, which is thus *proved to be a sufficient as well as a necessary condition* for the equivalence of the two expressions.

165. (ii) Secondly, let the invariantive integer of Δ_1 be $2r-2$, so that $p = 2r-1$. Then all the minors of order $2r$ of Δ_1 vanish; so that since they are, as before, the products of determinants of order $2r$ of the system

$$\ldots \left\| \frac{\partial z_1}{\partial x_s}, \ldots, \frac{\partial z_r}{\partial x_s}, \frac{\partial Z_1}{\partial x_s}, \ldots, \frac{\partial Z_r}{\partial x_s} \right\| \ldots \qquad (s = 1, 2, \ldots, n)$$

by determinants of the same order of the same system differently arranged, all the determinants of order $2r$ of this system vanish. Hence there is one identical functional relation among the quantities z and Z; and, since the quantities z are independent of one another, at least one of the quantities Z must occur in the relation, which may thus be taken in the form

$$\phi(z_1, \ldots, z_r, Z_1, \ldots, Z_r) = 0.$$

By the invariance of the two integers, $2r-2$ and $2r$ associated respectively with Δ_1 and with Δ_2, it follows that an expression $\sum_{s=1}^{n} X_s' dx_s'$, which is derived from the expression $\sum_{s=1}^{n} X_s dx_s$ by the (unknown) equations of transformation, can be expressed in the form

$$\sum_{i=1}^{r} Z_i' dz_i',$$

where the r quantities z' are independent of one another, and one identical functional relation subsists among the $2r$ quantities z' and Z', which must involve at least one of the quantities Z'. The functions X' not necessarily (nor generally) being the same functions of the variables x' as X are of the variables x, this relation is

not necessarily of the same form as the previous one, and it will in general be different, say of the form

$$\psi(z_1', \ldots, z_r', Z_1', \ldots, Z_r') = 0,$$

where ψ is different from ϕ.

The difference of these functional relations prevents the system of equations

$$z_i = z_i', \; Z_i = Z_i' \qquad (i = 1, \ldots, r),$$

which would transform $\Sigma Z_i dz_i$ into $\Sigma Z_i' dz_i'$, from being a consistent system*; and therefore the process, which is effective when the single invariant p is even, does not necessarily prove effective when the single invariant p is odd.

166. In order then to obtain an effective process for the case of a single invariantive odd integer, we reduce the case to that of an invariantive integer which is the next lower even number: this will require that the invariantive integer to be associated with Δ_1 shall be $2r-2$, the same as before, and that the integer to be associated with Δ_2 shall be $2r-2$, less than that which we had before. Hence Δ_1 may be left unchanged, and Δ_2 must be changed.

Adequate changes will be obtained, if we change the coefficients X by such decrements as shall leave the quantities a_{ij} unaltered; and so we replace X_i by a new quantity

$$X_i - \frac{\partial z}{\partial x_i},$$

where z is a function of $x_1, \ldots, x_n$. The determinant Δ_1 is not altered by this modification; and therefore the invariantive integer associated with it is still $2r-2$. We determine the introduced function z so that the integer associated with the modified Δ_2, say with ∇_2, in the form

$$\begin{vmatrix} a_{11}, & \ldots\ldots\ldots\ldots, & a_{1n}, & X_1 - \dfrac{\partial z}{\partial x_1} \\ \multicolumn{4}{c}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots} \\ a_{n1}, & \ldots\ldots\ldots\ldots, & a_{nn}, & X_n - \dfrac{\partial z}{\partial x_n} \\ -X_1 + \dfrac{\partial z}{\partial x_1}, & \ldots, & -X_n + \dfrac{\partial z}{\partial x_n}, & 0 \end{vmatrix}$$

* The system ceases to be inconsistent, if ϕ and ψ be the same functions. In particular, this occurs when $Z_m = 1$, and $Z_m' = 1$; see § 126.

shall be $2r-2$, being the even integer next lower than that associated with Δ_2.

The effect of this change on the expression $\sum_{i=1}^{n} X_i dx_i$ is to replace it by

$$\sum_{i=1}^{n}\left(X_i - \frac{\partial z}{\partial x_i}\right) dx_i,$$

that is, by

$$\sum_{i=1}^{n} X_i dx_i - dz;$$

and then, when z has been determined in accordance with the preceding condition, the single invariantive integer of the new expression is even and equal to $2(r-1)$, so that by the preceding result we have

$$\sum_{i=1}^{n} X_i dx_i - dz = \sum_{s=1}^{r-1} Y_s dy_s,$$

where the $2(r-1)$ quantities y and Y are independent of one another; and this is the smallest number of quantities which can occur on the right-hand side. As in § 162,

$$X_i - \frac{\partial z}{\partial x_i} = \sum_{s=1}^{r-1} Y_s \frac{\partial y_s}{\partial x_i},$$

and

$$a_{ij} = \sum_{s=1}^{r-1}\left(\frac{\partial Y_s}{\partial x_j}\frac{\partial y_s}{\partial x_i} - \frac{\partial Y_s}{\partial x_i}\frac{\partial y_s}{\partial x_j}\right).$$

Hence the minors of order $2r$ of Δ_2, which is

$$\begin{vmatrix} a_{11}, \ldots\ldots\ldots\ldots, & a_{1n}, & X_1 \\ \ldots\ldots\ldots\ldots\ldots\ldots & & \\ a_{n1}, \ldots\ldots\ldots\ldots, & a_{nn}, & X_n \\ -X_1, \ldots\ldots\ldots, & -X_n, & 0 \end{vmatrix},$$

are obtained by multiplying some one determinant of order $2r$ of the system

$$\cdots \left\Vert \begin{matrix} 0, & \dfrac{\partial z}{\partial x_1}, & \dfrac{\partial y_1}{\partial x_1}, \ldots, & \dfrac{\partial y_{r-1}}{\partial x_1}, & \dfrac{\partial Y_1}{\partial x_1}, \ldots, & \dfrac{\partial Y_{r-1}}{\partial x_1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0, & \dfrac{\partial z}{\partial x_n}, & \dfrac{\partial y_1}{\partial x_n}, \ldots, & \dfrac{\partial y_{r-1}}{\partial x_n}, & \dfrac{\partial Y_1}{\partial x_n}, \ldots, & \dfrac{\partial Y_{r-1}}{\partial x_n} \\ 1, & 0, & 0, \ldots, & 0, & Y_1, \ldots, & Y_{r-1} \end{matrix} \right\Vert \cdots$$

by some one determinant of order $2r$ of the system

$$\cdots\left\|\begin{array}{cccccccc} -\dfrac{\partial z}{\partial x_1}, & 0, & \dfrac{\partial Y_1}{\partial x_1}, & \ldots, & \dfrac{\partial Y_{r-1}}{\partial x_1}, & -\dfrac{\partial y_1}{\partial x_1}, & \ldots, & -\dfrac{\partial y_{r-1}}{\partial x_1} \\ \multicolumn{8}{c}{\dotfill} \\ -\dfrac{\partial z}{\partial x_n}, & 0, & \dfrac{\partial Y_1}{\partial x_n}, & \ldots, & \dfrac{\partial Y_{r-1}}{\partial x_n}, & -\dfrac{\partial y_1}{\partial x_n}, & \ldots, & -\dfrac{\partial y_{r-1}}{\partial x_n} \\ 0, & 1, & Y_1, & \ldots, & Y_{r-1}, & 0, & \ldots, & 0 \end{array}\right\|\cdots$$

the same as the other but differently arranged. Now since $2r$ is the invariantive integer associated with Δ_2, so that the minors of order $2r$ do not all vanish, it follows that the determinants of order $2r$ of the foregoing repeated system do not all vanish; and therefore the functions $z, y_1, \ldots, y_{r-1}, Y_1, \ldots, Y_{r-1}$ are independent of one another, being functions such that

$$\sum_{i=1}^{n} X_i dx_i = dz + \sum_{s=1}^{r-1} Y_s dy_s.$$

On account of the persistence of the invariantive integers, $2r-2$ and $2r$ associated with Δ_1 and Δ_2 respectively, it follows that an expression $\sum_{i=1}^{n} X_i' dx_i'$, which is derived from $\sum_{i=1}^{n} X_i dx_i$ by the (unknown) equations of transformation, can be expressed in the form

$$dz' + \sum_{s=1}^{r-1} Y_s' dy_s',$$

where the $2r-1$ quantities $z', y_1', \ldots, y'_{r-1}, Y_1', \ldots, Y'_{r-1}$ are independent of one another.

Since the $2r-1$ quantities z, y, Y are independent of one another, as are also the $2r-1$ quantities z', y', Y', the simplest equations of relation which will transform $dz + \sum_{i=1}^{r-1} Y_i dy_i$ into $dz' + \sum_{i=1}^{r-1} Y_i' dy_i'$ are

$$z = z'; \; y_i = y_i', \; Y_i = Y_i' \qquad (i = 1, 2, \ldots, r-1).$$

Substituting for z, y, Y, z', y', Y' their respective values in terms of x and of x', we have equations of relation which are sufficient to give the equation

$$\sum_{i=1}^{n} X_i dx_i = \sum_{i=1}^{n} X_i' dx_i'$$

and these equations have been obtained on the assumption of the *invariantive persistence of the odd integer* $p = 2r - 1$, which is thus *proved to be a sufficient as well as a necessary condition* for the equivalence of the two expressions.

Hence the theorem enunciated in § 158—that the equivalence of the expressions is established by the existence of the invariantive integer p—is proved.

167. On account of the fundamental importance of this integer, Frobenius classifies the Pfaffian expressions according to the integer associated with them. *An expression, which has p for its invariantive integer, is said to be of the p^{th} class*; the preceding investigation shews that there are p independent functions in the reduced form equivalent to the expression, which is

$$\sum_{i=1}^{\frac{1}{2}p} Z_i dz_i \quad \text{or} \quad dz + \sum_{i=1}^{\frac{1}{2}(p-1)} Y_i dy_i,$$

according as p is even or uneven*.

168. Though the present method is concerned chiefly with the general theory of the transformation of Pfaffian expressions into one another, and not primarily with their reduction to a normal form and the thence inferred system of integrals, yet the normal form enters as an essential part of the investigation by leading to the formation of the minimum number of independent equations which render the transformation possible. No novel method is given for the solution of the partial differential equations which determine the differential elements of the normal form; the following summary, however, of the derivation of those equations may be convenient.

Let $2r$ be the highest order of non-vanishing minors of the determinant

$$\Delta_2 = \begin{vmatrix} a_{11}, \ldots\ldots, & a_{1n}, & X_1 \\ \ldots\ldots\ldots & \ldots\ldots & \ldots \\ a_{n1}, \ldots\ldots, & a_{nn}, & X_n \\ -X_1, \ldots\ldots, & -X_n, & 0 \end{vmatrix};$$

* See note to § 158.

then either $2r$ or $2r-2$ is the highest order of non-vanishing minors of the determinant

$$\Delta_1 = \begin{vmatrix} a_{11}, & \ldots\ldots, & a_{1n} \\ \ldots & \ldots & \ldots \\ a_{n1}, & \ldots\ldots, & a_{nn} \end{vmatrix}.$$

The arithmetic mean p of the integers associated with the two determinants is an invariant, which determines the class of the Pfaffian expression $\sum_{i=1}^{n} X_i dx_i$; in one case p is even and equal to $2r$, in the other p is odd and equal to $2r-1$.

In the case when $p = 2r$, the normal form is

$$\sum_{i=1}^{r} Z_i dz_i;$$

the quantities Z are determined from any r independent equations of the set

$$X_s = \sum_{i=1}^{r} Z_i \frac{\partial z_i}{\partial x_s},$$

and the quantities z are determined by the r sets of partial differential equations, which express the conditions that the minors of order $2r+2$ in the determinants

$$\begin{vmatrix} a_{11}, & \ldots\ldots, & a_{1n}, & X_1, & \frac{\partial z_1}{\partial x_1}, & \ldots\ldots, & \frac{\partial z_k}{\partial x_1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1}, & \ldots\ldots, & a_{nn}, & X_n, & \frac{\partial z_1}{\partial x_n}, & \ldots\ldots, & \frac{\partial z_k}{\partial x_n} \\ -X_1, & \ldots\ldots, & -X_n, & 0, & 0, & \ldots\ldots, & 0 \\ -\frac{\partial z_1}{\partial x_1}, & \ldots\ldots, & -\frac{\partial z_1}{\partial x_n}, & 0, & 0, & \ldots\ldots, & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ -\frac{\partial z_k}{\partial x_1}, & \ldots\ldots, & -\frac{\partial z_k}{\partial x_n}, & 0, & 0, & \ldots\ldots, & 0 \end{vmatrix}$$

all vanish; the successive quantities z being obtained by assigning to k in succession the values $1, \ldots, r$. Not more than r quantities z can be thus determined; and the $2r$ quantities Z and z are functionally independent of one another.

In the case when $p = 2r - 1$, the normal form is

$$dz + \sum_{i=1}^{r-1} Y_i dy_i;$$

the quantities Y are determined from any $r - 1$ independent equations of the set

$$X_s = \frac{\partial z}{\partial x_s} + \sum_{i=1}^{r-1} Y_i \frac{\partial y_i}{\partial x_s};$$

the quantity z is determined by the set of partial differential equations, which express the conditions that the minors of order $2r$ in the determinant

$$\begin{vmatrix} a_{11} & ,\ldots\ldots, & a_{1n} & , X_1 - \dfrac{\partial z}{\partial x_1} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & ,\ldots\ldots, & a_{nn} & , X_n - \dfrac{\partial z}{\partial x_n} \\ -X_1 + \dfrac{\partial z}{\partial x_1}, & \ldots\ldots, & -X_n + \dfrac{\partial z}{\partial x_n}, & 0 \end{vmatrix}$$

all vanish; and the quantities y are determined by the $r - 1$ sets of partial differential equations, which express the conditions that the minors of order $2r$ in the determinant

$$\begin{vmatrix} a_{11} & ,\ldots\ldots, & a_{1n} & , X_1 - \dfrac{\partial z}{\partial x_1}, & \dfrac{\partial y_1}{\partial x_1}, & \ldots\ldots, & \dfrac{\partial y_k}{\partial x_1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & ,\ldots\ldots, & a_{nn} & , X_n - \dfrac{\partial z}{\partial x_n}, & \dfrac{\partial y_1}{\partial x_n}, & \ldots\ldots, & \dfrac{\partial y_k}{\partial x_n} \\ -X_1 + \dfrac{\partial z}{\partial x_1}, & \ldots\ldots, & -X_n + \dfrac{\partial z}{\partial x_n}, & 0 & , 0 & ,\ldots\ldots, & 0 \\ -\dfrac{\partial y_1}{\partial x_1} & ,\ldots\ldots, & -\dfrac{\partial y_1}{\partial x_n} & , 0 & , 0 & ,\ldots\ldots, & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ -\dfrac{\partial y_k}{\partial x_1} & ,\ldots\ldots, & -\dfrac{\partial y_k}{\partial x_n} & , 0 & , 0 & ,\ldots\ldots, & 0 \end{vmatrix}$$

all vanish; the successive quantities y being obtained by assigning to k in succession the values $1, \ldots, r - 1$. Not more than $r - 1$

quantities y can be thus determined; and the $2r-1$ quantities z, y, Y are functionally independent of one another.

Ex. 1. The following results given by Frobenius* are easily obtainable:—

(*a*) If the class of a Pfaffian differential expression be altered in consequence of multiplication of the expression by a factor, the change is an increase or a decrease of one unit according as the class is odd or even.

(*b*) If the class of a Pfaffian differential expression be altered in consequence of addition to the expression of a perfect differential, the change is a decrease or an increase of one unit according as the class is odd or even.

(*c*) Hence deduce the effect produced on the class of an unconditioned Pfaffian differential expression by (i) multiplying it by a factor (ii) increasing it by the addition of a perfect differential.

Ex. 2. The coefficients of the differential elements in the expression

$$X_1dx_1 + \dots\dots + X_ndx_n$$

are homogeneous functions of the μ^{th} degree; and X denotes the homogeneous function

$$X_1x_1 + \dots\dots + X_nx_n.$$

Shew that, if X be a constant different from zero, then the class of the original expression is odd; that, if X be zero and μ not equal to -1, the class is even; and that, if X be zero, the expression can be a perfect differential only if μ be -1.

(Frobenius.)

Ex. 3. Shew, with the notation of the last question, that, if X vanish, then the class of the Pfaffian expression is less than n, if either n be odd or μ be -1.

Shew also that, if X vanish, while μ is not equal to -1 and the expression is unconditioned, its class is diminished by one unit, if the expression be divided by an arbitrary homogeneous function of degree $\mu+1$.

(Frobenius.)

Ex. 4. If n functions, given by the equations

$$A_i(f) = \sum_{j=1}^{n} A_{ij}\frac{\partial f}{\partial x_j} \qquad (i=1, \dots\dots, n)$$

independent of one another, be such as to satisfy the relations

$$A_s\{A_i(f)\} = A_i\{A_s(f)\},$$

then

$$\begin{vmatrix} A_{11}, \dots\dots\dots\dots, A_{1n} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ A_{n-1,1}, \dots\dots, A_{n-1,n} \\ dx_1, \dots\dots\dots\dots, dx_n \end{vmatrix} \div \begin{vmatrix} A_{11}, \dots\dots\dots\dots\dots, A_{1n} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ A_{n1}, \dots\dots\dots\dots\dots, A_{nn} \end{vmatrix}$$

is a perfect differential, the n^2 quantities A being functions of the variables x subject only to the indicated limitations.

(Frobenius.)

* *Crelle*, t. lxxxvi. (1879), pp. 1—19.

CHAPTER XII.

ABSTRACT OF DARBOUX'S METHOD.

THE investigations of M. Darboux on Pfaff's problem are contained in a memoir* published in 1882, though most of it which bears directly on the theory was written in 1876. He deals more with the theory of the forms than with the methods of integration of differential equations which occur in the theory; in this respect he resembles Frobenius. Moreover in his process there is a certain similarity to that adopted by Frobenius, for the basis is the invariantive character of certain expressions, in particular, of the associated bilinear covariant: and therefore in point of publication Darboux has been considerably anticipated by Frobenius. The remainder of the memoir deals with the theory of the tangential transformation: and, though the method adopted for it is distinct, the results of the theory were already known from the earlier published memoirs of Lie and Mayer.

Under these circumstances I shall state merely the results, without proof, so as to give an indication of the course of the memoir.

169. (i) He takes

$$\Theta_d = \sum_{i=1}^{n} X_i dx_i, \qquad \Theta_\delta = \sum_{i=1}^{n} X_i \delta x_i,$$

and proves

$$\delta\Theta_d - d\Theta_\delta = \Sigma\Sigma\, a_{ik}\, dx_i\, \delta x_k,$$

which, being independent in value of the particular set of variables, is an invariant for change of variables.

* "Sur le problème de Pfaff," *Comptes Rendus*, t. xciv. (1882), pp. 835—837 *Darb. Bull.*, 2^me^ Sér. t. vi. (1882), pp. 14—36, 49—68.

(ii) The first step towards the construction of the normal form is the choice of new independent variables such that, when Θ_d is expressed in terms of them, there shall be fewer differential elements. For this purpose the subsidiary system of equations

$$\begin{gathered} a_{11}\,dx_1 + \ldots + a_{n1}\,dx_n = \lambda X_1\,dt \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{1n}\,dx_1 + \ldots + a_{nn}\,dx_n = \lambda X_n\,dt \end{gathered}$$

is used. It is an invariantive system: and it can be replaced by the single equation

$$\delta\Theta_d - d\Theta_\delta = \lambda\Theta_\delta\,dt,$$

supposed true whatever be the variations δx.

(iii) When n is even and the determinant Δ of the coefficients of the quantities dx in the subsidiary system is not zero, then the $n-1$ integrals $y_1, \ldots, y_{n-1}$ of the equations, which are independent of t, are taken as new independent variables; and with them is associated an arbitrary function y_n of the original variables, so that $y_1, \ldots, y_n$ are n independent variables. Then by the invariantive property of the subsidiary equations it is easily proved that

$$\Theta_d = y_n \sum_{r=1}^{n-1} Y_r^0\,dy_r,$$

where the coefficients Y_r^0 are independent of y_n.

(iv) When n is odd, so that Δ necessarily vanishes, then, if the first minors do not all vanish, λ is taken to be zero in the subsidiary system; the ratios of the differential elements are determinate, and $n-1$ integrals of the consequent equations are obtainable. If these integrals be denoted by $y_1, \ldots, y_{n-1}$, then, when they are associated with a new arbitrary and independent function y_n, we may take the n quantities as a set of n independent variables. It is proved that

$$\Theta_d = d\Psi + \sum_{r=1}^{n-1} Y_r^0\,dy_r,$$

where the coefficients Y_r^0 are independent of y_n.

Two cases occur. If Ψ contain y_n, it may itself be taken as y_n, so that

$$\Theta_d = dy_n + \sum_{r=1}^{n-1} Y_r^0\,dy_r.$$

If Ψ be independent of y_n, then Θ_d can be expressed in the form

$$\Theta_d = \sum_{r=1}^{n-1} Y_r^{(0)} dy_r.$$

(v) Should the subsidiary system, even with the limitation $\lambda = 0$, not be determinate but contain only p distinct equations, then Darboux makes it determinate, as in the preceding cases, by the association with it of $n-p-1$ equations

$$d\phi_1 = 0, \ \ldots\ldots, \ d\phi_{n-p-1} = 0.$$

The argument proceeds as before; and the general result of the first transformation is that an expression Θ_d can always be changed into one or other of the three forms

$$y_n \sum_{r=1}^{n-1} Y_r\, dy_r,$$

$$\sum_{r=1}^{n-1} Y_r\, dy_r,$$

$$dy_n + \sum_{r=1}^{n-1} Y_r\, dy_r,$$

the variables $y_1, \ldots, y_{n-1}, y_n$ being independent and the coefficients Y depending only on the variables $y_1, \ldots, y_{n-1}$.

(vi) Hence an expression Θ_d can always be brought to one or other of the types

$$dy - \sum_{r=1}^{p} z_r\, dy_r,$$

$$\sum_{r=1}^{p} z_r\, dy_r,$$

where the quantities y, z_i, y_i are functions of all the variables in Θ_d and are independent of one another; and $2p+1$ or $2p$, according as the type is the first or the second, is not greater than n.

The former or the latter is the type to which Θ_d can be reduced, according as the subsidiary equations cannot or can be satisfied by taking λ different from zero. The integer p is determined as one-half of the number (necessarily even) of independent equations in the subsidiary system, and the $2p$ quantities y_i and z_i are a set of integrals of these independent equations. For the former type, the quantities y and z are necessarily independent of λ, which has been made zero for the associated subsidiary system;

for the latter type, the ratios of the quantities z are independent of λ.

(vii) Darboux adopts Cauchy's method of integration of the subsidiary equations; and he is led to results, which are similar to Lie's (§§ 147, 154), relating to the transformation of the expression to an equivalent unconditioned form; they are as follows.

(*a*) If the canonical form of the expression be

$$\sum_{r=1}^{p} z_r\, dy_r,$$

then the subsidiary system, consisting of $2p$ independent equations, has $2p-1$ integrals independent of λ. There must thus be at least $n-2p+1$ variables, say $x_{2p}, \ldots, x_n$, which are not integrals of the system. If then the $2p-1$ integrals be taken in the form of principal integrals and be denoted by $u_1, \ldots, u_{2p-1}$, so that $u_1, \ldots, u_{2p-1}$ reduce to $x_1, \ldots, x_{2p-1}$ respectively when constant values $a_{2p}, \ldots, a_n$ are assigned to $x_{2p}, \ldots, x_n$, the equivalent unconditioned form is

$$\Theta_d = K \sum_{r=1}^{2p-1} U_r du_r,$$

where U_r is the value of X_r, when x_s is replaced by u_s for $s = 1, \ldots, 2p-1$ and by a_s for $s = 2p, \ldots, n$.

(*b*) If the canonical form of the expression be

$$dy - \sum_{r=1}^{p} z_r\, dy_r,$$

then $\lambda = 0$ in the subsidiary system, and the set of $2p$ independent equations has $2p$ independent integrals. There must thus be at least $n-2p$ variables, say $x_{2p+1}, \ldots, x_n$, which are not integrals of the set. If the $2p$ integrals be taken in the form of principal integrals and be denoted by $u_1, \ldots, u_{2p}$, so that $u_1, \ldots, u_{2p}$ reduce to $x_1, \ldots, x_{2p}$ respectively when constant values $a_{2p+1}, \ldots, a_n$ are assigned to $x_{2p+1}, \ldots, x_n$, the equivalent unconditioned form is

$$\Theta_d = dH + \sum_{r=1}^{2p} U_r\, du_r,$$

where U_r is the value of X_r, when x_s is replaced by u_s for $s = 1, \ldots, 2p$ and by a_s for $s = 2p+1, \ldots, n$, and H is a function which vanishes for the last set of substitutions of constants for the $n-2p$ variables x.

170. These results are applied to the solution of partial differential equations of the first order (§ 136).

The remainder of the memoir is devoted to an investigation of the relations which are the basis of Clebsch's method of integration (§ 125) and to the establishment* of the theory of tangential transformations proceeding (as in the earlier part of the memoir) from two simultaneous sets of variations. It is superfluous to reproduce the investigation here; the results agree with those earlier established by Lie and independently by Mayer and already discussed in Chapter IX.

Ex. Prove that, with the ordinary notation used in connection with Pfaffian differentials, the two expressions

$$\begin{vmatrix} a_{11}, \dots\dots\dots, a_{n1}, P_1 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{1n}, \dots\dots\dots, a_{nn}, P_n \\ \dfrac{\partial\phi}{\partial x_1}, \dots\dots\dots, \dfrac{\partial\phi}{\partial x_n}, 0 \end{vmatrix} \div \begin{vmatrix} a_{11}, \dots\dots\dots\dots\dots, a_{n1} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{1n}, \dots\dots\dots\dots\dots, a_{nn} \end{vmatrix},$$

obtained by taking, first, $P_1, \dots, P_n = X_1, \dots, X_n$ and, secondly, $P_1, \dots, P_n = \dfrac{\partial\psi}{\partial x_1}, \dots, \dfrac{\partial\psi}{\partial x_n}$, are invariants for change of the independent variables, the Pfaffian expression being supposed unconditioned.

Shew that these expressions are the quantities denoted by Clebsch by the symbols (ϕ) and $(\phi\psi)$ in his theory of Pfaff's problem.

(Darboux.)

* This part of his investigation is followed by Jordan, *Cours d'Analyse*, t. iii. (1887), pp. 339—348.

CHAPTER XIII.

SYSTEMS OF PFAFFIANS.

IT will be seen that in the theory of systems of unconditioned Pfaffians hardly any advances have been made. In fact, there are very few investigations which deal with systems of simultaneous non-exact equations; and even those which are published discuss for the most part such exact integrals as the systems may possess. The following are the principal sources of information on the subject:—

BIERMANN; "Ueber n simultane Differentialgleichungen der Form $\Sigma X_\mu dx_\mu = 0$," *Schlöm. Zeitschrift*, t. xxx. (1885), pp. 234—244.

BOOLE; "On simultaneous differential equations of the first order in which the number of the variables exceeds by more than one the number of the equations," *Phil. Trans.*, 1862, pp. 437—454.

"On the differential equations of dynamics," *Phil. Trans.*, 1863, pp. 485—501.

Supplementary volume of *Treatise on Differential Equations*, 1865, pp. 74—89.

ENGEL; "Zur Invariantentheorie der Systeme von Pfaffschen Gleichungen," *Leipz. Sitzungsb.*, (1889), pp. 157—176, *ib.*, (1890), pp. 192—207; this connects itself with §§ 128, 129 of Lie's *Theorie der Transformationsgruppen* (Leipzig, 1888), which deal with the character of the transformations of which such a system admits.

FROBENIUS; "Ueber das Pfaff'sche Problem," *Crelle*, t. lxxxii. (1877); especially pp. 287—289 of the memoir.

IMSCHENETSKY; "Intégration des équations aux dérivées partielles du second ordre d'une fonction de deux variables," *Grun. Arch.*, t. liv. (1872); specially pp. 290—314 of the memoir.

TANNER; "Preliminary note on a generalisation of Pfaff's problem," *Lond. Math. Soc. Proc.*, vol. xi. (1880), pp. 131—139*.

VOSS; "Ueber die Differentialgleichungen der Mechanik," *Math. Ann.*, t. xxv. (1885); especially pp. 258—263 of the memoir.

* For various reasons, I am unable to accept the results obtained by Prof. Tanner in this paper.

171. The system of n equations linear in the differential elements of the variables (and the coextensive system of associated partial differential equations linear in the derivatives of the dependent function), which were discussed in Chapter II., subsisted as a simultaneous exact system on the hypothesis that the conditions given in § 26 as the criteria of exactness were satisfied: and on this hypothesis there was inferred the existence of an equivalent system of n integral equations of the type $u_r = c_r$. Each such equation is, in and by itself, an integral of the system of differential equations; that is to say, restricting ourselves to the system of n equations, the inferred equation

$$du_r = 0$$

is completely satisfied by means of the original differential equations, there being no need to use for the purpose of satisfying it any of the other integral equations: and such an equation as

$$du_r = 0$$

is merely a linear combination of some (or all) of the given differential equations.

When the whole system of n integrals is considered simultaneously (each of them being, in and by itself, an integral of the system), we obtain n equations

$$du_1 = 0, \; \ldots\ldots, \; du_n = 0;$$

each of these is satisfied in virtue of the original equations and is a linear combination of those equations. But the n integrals are independent so that the n equations $du = 0$ are independent: that is, the linear combinations of the foregoing equations are independent of one another, and therefore the original equations can be derived from the equations $du = 0$ which are an immediate consequence of the integral system. Hence the system of n inferred integrals and the system of n original differential equations are completely equivalent to and coextensive with one another; provided that, as already remarked, the conditions of Chapter II.—there proved sufficient and necessary—be all satisfied. The system is said in that case to be *completely integrable*.

But the system of ordinary differential equations will still subsist as a simultaneous system, when only some, or even when

none, of the conditions of integrability are satisfied. In the former case, when some of the specified conditions are satisfied, the system of equations may have some exact integrals; that is to say, there may be some equations of the form $u = c$ such that

$$du = 0$$

is satisfied solely by means of the system of differential equations without reference to any integral relations among the variables. The number of these exact integrals must be less than n, otherwise the system would be completely integrable: and each of them (necessarily supposed to be independent) leads, when differentiated, to a linear combination of the system of equations, the various combinations being independent. When such a system possesses a number of exact integrals, the number being less than the number of equations, it is said to be *incompletely integrable.*

It is evident, after the earlier explanation, that the set of exact integrals possessed by an incompletely integrable system is not an equivalent of the system of differential equations. In fact, as each member of the set leads to a linear combination of the system of equations, we have a number of independent linear combinations of the members of the system less than the number of members; and therefore the system of differential equations cannot be inferred from the exact integrals. The supply of this deficiency from an integral equivalent of the differential equations, and the use of the exact integrals for the modification of the differential system, will be subsequently discussed.

Lastly, it may happen that the given system of equations possesses no exact integral, that is, that there is no linear combination of the equations which can lead to an equation

$$du = 0.$$

In this case the system is said to be *non-integrable*: and, as before, a question as to its integral equivalent will arise.

172. The integration of a completely integrable system has already been discussed: we proceed to the discussion of the characteristics of an incompletely integrable system. Naturally, the first step is the determination of the number and the form of the exact integrals.

We take the system of n equations, involving $m+n$ variables, in the form

$$\left.\begin{array}{l}\Omega_1 = -dx_{m+1} + A_{11}dx_1 + A_{12}dx_2 + \ldots\ldots + A_{1m}dx_m = 0\\ \Omega_2 = -dx_{m+2} + A_{21}dx_1 + A_{22}dx_2 + \ldots\ldots + A_{2m}dx_m = 0\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \Omega_n = -dx_{m+n} + A_{n1}dx_1 + A_{n2}dx_2 + \ldots\ldots + A_{nm}dx_m = 0\end{array}\right\} \text{(I)},$$

the coefficients A being functions of the variables x.

Let $\phi = c$ be an exact integral of the system (I); then the differential equation

$$d\phi = 0$$

is a linear combination of the n equations $\Omega = 0$ and is satisfied in virtue of those equations. The form of the combination is obvious, being

$$d\phi = -\sum_{r=1}^{n} \frac{\partial\phi}{\partial x_{m+r}} \Omega_r;$$

and so we have, after substituting the full expression for $d\phi$ and transposing the right-hand side, the equation

$$\sum_{s=1}^{m} \frac{\partial\phi}{\partial x_s} dx_s + \sum_{r=1}^{n} \frac{\partial\phi}{\partial x_{m+r}} \left(\sum_{s=1}^{m} A_{rs} dx_s\right) = 0,$$

which is satisfied in connection with (I). But this equation involves only the differential elements $dx_1, \ldots, dx_m$, among which no relation is given by (I); and therefore the coefficient of each element must vanish, so that we have the m equations

$$\Delta_s\phi = \frac{\partial\phi}{\partial x_s} + \sum_{r=1}^{n} A_{rs} \frac{\partial\phi}{\partial x_{m+r}} = 0 \quad \ldots\ldots\ldots\ldots\ldots\text{(II)}$$

(for $s = 1, \ldots, m$) satisfied for an exact integral ϕ of the given system (I); and they must be satisfied by every such integral.

Now this system (II) is in form the same as the corresponding system (II) of § 38; but the present one is not complete (in the Jacobian sense), for the conditions for coexistence and the possession of common solutions are not all satisfied, as they are for the earlier system. Thus the quantities

$$(\Delta_i, \Delta_j) = (\Delta_i\Delta_j - \Delta_j\Delta_i)\,\phi$$

do not all vanish in virtue of (II); and so we may have new non-vanishing expressions $\nabla_s\phi$, all linear and homogeneous in the

partial first differential coefficients of ϕ with regard to $x_{m+1}, \ldots, x_{m+n}$. In order that functions ϕ of the kind indicated may exist, the ordinary theory requires that

$$\nabla_s \phi = 0$$

for each non-evanescent operator. Hence new differential equations are introduced into the system: and we must continue the application of the Jacobian conditions in the form

$$(\Delta_s, \nabla_t) = 0, \qquad (\nabla_s, \nabla_t) = 0,$$

retaining every non-evanescent and unsatisfied condition as a new equation to be combined with the system already obtained, until no new equations are thus formed. The system, thus increased, is now a complete system: and the members of the system are linearly independent of one another.

The original system (II) contained m equations: let the members, necessary to make it a complete system, be p in number, say

$$\nabla_s \phi = \sum_{r=1}^{n} B_{rs} \frac{\partial \phi}{\partial x_{m+r}} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(III)}$$

for $s = 1, \ldots, p$: then the function ϕ is a solution of the $m+p$ simultaneous linear homogeneous partial differential equations of the complete system constituted by (II) and (III).

If *p be less than n*, then the complete system of $m+p$ equations involving $m+n$ variables has, by § 38, $n-p$ functionally independent solutions; and therefore the original system (I) of differential equations has $n-p$ exact integrals.

If *p be equal to n*, then since the equations (II) and (III) are linearly independent and are linearly homogeneous in the $m+n$ quantities $\dfrac{\partial \phi}{\partial x}$, they can be satisfied only by having each of the derivatives of ϕ zero; and therefore ϕ itself must be a mere constant, a result nugatory so far as concerns exact integrals.

Similarly, if *p be greater than n*, the complete system can be satisfied only by zero values of the derivatives of ϕ, which also lead to a result nugatory so far as concerns the possession of exact integrals.

Hence it appears that *the given system (I) possesses exact integrals only if the associated system (II) of partial differential equa-*

tions can be rendered complete by the addition of fewer than n new equations; and if for this purpose p new equations must be added, then the number of exact integrals is n − p.

173. Thus the conditions for the possession of $n-p$ exact integrals are the conditions that the system (II) of associated partial equations should be rendered complete by the addition to them of p derived equations. The express form of these conditions for the most general case can be inferred from the form of the conditions for the following particular case.

Consider the equations

$$\left.\begin{aligned} du &= U_1\,dx_1 + U_2\,dx_2 + U_3\,dx_3 + U_4\,dx_4 \\ dv &= V_1\,dx_1 + V_2\,dx_2 + V_3\,dx_3 + V_4\,dx_4 \\ dw &= W_1dx_1 + W_2dx_2 + W_3dx_3 + W_4dx_4 \end{aligned}\right\}$$

as a system; the associated partial equations which correspond to (II) are

$$\Delta_r\phi = \frac{\partial\phi}{\partial x_r} + U_r\frac{\partial\phi}{\partial u} + V_r\frac{\partial\phi}{\partial v} + W_r\frac{\partial\phi}{\partial w} = 0$$

for $r=1, 2, 3, 4$. Let

$$\Delta_i P_j - \Delta_j P_i = P_{ij}$$

(for $P = U, V, W$); then the conditions for the coexistence of the four equations and for the possession of common solutions are the six equations

$$\Delta_{ij}\phi = U_{ij}\frac{\partial\phi}{\partial u} + V_{ij}\frac{\partial\phi}{\partial v} + W_{ij}\frac{\partial\phi}{\partial w} = 0,$$

some of which must neither be evanescent nor be satisfied in virtue of the earlier system. Again, let

$$\Delta_k P_{ij} - \Delta_{ij} P_k = {}_kP_{ij}$$

for any symbol k; let

$$\Delta_{\theta k} P_{ij} - {}_k\Delta_{ij} P_\theta = {}^{\theta}_{k}P_{ij}$$

for any symbol θ, and so on: and let the rectangular array

$$\cdots\left\|\begin{matrix} U_{ij}\,, & {}_kU_{ij}\,, & {}_{lk}U_{ij}\,, & {}^{m}_{lk}U_{ij}, & {}^{mn}_{lk}U_{ij}, & {}^{mn}_{lk}U^{p}_{ij}, & {}^{mn}_{lk}U^{pq}_{ij} \\ V_{ij}\,, & {}_kV_{ij}\,, & {}_{lk}V_{ij}\,, & {}^{m}_{lk}V_{ij}, & {}^{mn}_{lk}V_{ij}, & {}^{mn}_{lk}V^{p}_{ij}, & {}^{mn}_{lk}V^{pq}_{ij} \\ W_{ij}, & {}_kW_{ij}, & {}_{lk}W_{ij}, & {}^{m}_{lk}W_{ij}, & {}^{mn}_{lk}W_{ij}, & {}^{mn}_{lk}W^{p}_{ij}, & {}^{mn}_{lk}W^{pq}_{ij} \end{matrix}\right\|\cdots$$

for all values 1, 2, 3 of i, j, k, l, m, n, p, q be formed, there being seven types of quantities in the array.

Then in order that the original system may have *three* exact integrals (it cannot have more than three) independent of one another, it is necessary and sufficient that all the determinants of one constituent formed from the quantities of the type occurring in the first expressed column of the above array vanish: in other words that all the quantities U_{ij}, V_{ij}, W_{ij} vanish. These we shall call the conditions for three integrals.

In order that the original system may have only *two* exact integrals it is necessary and sufficient, first, that the conditions for three integrals shall not all be satisfied; second, that all the determinants of 2^2 constituents formed from the quantities of the types occurring in the first three expressed columns of the array shall vanish. These we shall call the conditions for two integrals; they secure that one new equation shall be added to the original system of four partial equations.

In order that the original system may have only *one* exact integral it is necessary and sufficient, first, that the conditions for two integrals shall not all be satisfied; second, that all the determinants of 3^2 constituents formed from all the quantities in the array shall vanish. These we shall call the conditions for one integral; they secure that two new equations shall be added to the original system of four partial equations.

Finally, in order that the original system may have *no* exact integral it is necessary and sufficient that not all the determinants of 3^2 constituents specified in the conditions for one integral shall vanish.

The generalisation to the system (I) is now evident. It would be necessary to form a rectangular array of n rows; the number of types of quantities occurring in the array would be 2^n-1; the number of quantities of any type would depend upon m and, after the first type*, also upon n. The conditions that the system should have only $n-p$ exact integrals are that the determinants of p^2 constituents formed from the quantities of type of the first 2^p-1 columns should vanish, but not all determinants of a smaller number of constituents.

Ex. 1. Infer (or otherwise prove) that the simultaneous system

$$\left.\begin{aligned} dx_3 &= x_4 dx_1 + x_5 dx_2 \\ dx_4 &= x_6 dx_1 + \theta dx_2 \\ dx_5 &= \theta dx_1 + x_7 dx_2 \end{aligned}\right\},$$

where θ is any function of all the seven variables, has no exact integral.

Ex. 2. The conditions that the two equations

$$\begin{aligned} dx_1 &= a_3 dx_3 + a_4 dx_4 \\ dx_2 &= \beta_3 dx_3 + \beta_4 dx_4 \end{aligned}$$

may have one exact integral or no exact integral are, first, that the quantities

$$\gamma_1 = A_3 a_4 - A_4 a_3, \quad \gamma_2 = A_3\beta_4 - A_4\beta_3,$$

where $A_3 = \dfrac{\partial}{\partial x_3} + a_3\dfrac{\partial}{\partial x_1} + \beta_3\dfrac{\partial}{\partial x_2}$ and $A_4 = \dfrac{\partial}{\partial x_4} + a_4\dfrac{\partial}{\partial x_1} + \beta_4\dfrac{\partial}{\partial x_2}$, do not both vanish; and, second, that the determinants of the second order in

$$\left\Vert \begin{matrix} \gamma_1, & A_3\gamma_1 - Ba_3, & A_4\gamma_1 - Ba_4 \\ \gamma_2, & A_3\gamma_2 - B\beta_3, & A_4\gamma_2 - B\beta_4 \end{matrix} \right\Vert,$$

where $B = \gamma_1\dfrac{\partial}{\partial x_1} + \gamma_2\dfrac{\partial}{\partial x_2}$, do all vanish or do not all vanish. (Engel.)

* The number of the first type is $\frac{1}{2}m(m-1)$.

Ex. 3. Consider the equations

$$\left.\begin{aligned} dz &= (t + xy + xz)\,dx + (xzt + y - xy)\,dy \\ dt &= (y + z - 3x)\,dx + (zt - y)\,dy \end{aligned}\right\},$$

a simultaneous system*. The associated partial differential equations are

$$\Delta\phi = \frac{\partial\phi}{\partial x} + (t + xy + xz)\frac{\partial\phi}{\partial z} + (y + z - 3x)\frac{\partial\phi}{\partial t} = 0,$$

$$\Delta'\phi = \frac{\partial\phi}{\partial y} + (xzt + y - xy)\frac{\partial\phi}{\partial z} + (zt - y)\frac{\partial\phi}{\partial t} = 0.$$

Forming the Jacobian condition we have, in the notation of the last example,

$$\gamma_1 = x(t^2 + xyt + xy + yz + z^2 - 3xz - 1 - y),$$
$$\gamma_2 = t^2 + xyt + xy + yz + z^2 - 3xz - 1 - y;$$

so that, rejecting the algebraical factor, the Jacobian condition is

$$\Delta''\phi = x\frac{\partial\phi}{\partial z} + \frac{\partial\phi}{\partial t} = 0.$$

The three equations $\Delta\phi = 0$, $\Delta'\phi = 0$, $\Delta''\phi = 0$ are easily proved to be a complete system; as they involve four independent variables, they have one integral.

The easiest way to obtain this integral is as follows. From the equations we have

$$\frac{\partial\phi}{\partial t} = -x\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial y} = -y\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = -(3x^2 + t)\frac{\partial\phi}{\partial z},$$

and therefore

$$d\phi = \frac{\partial\phi}{\partial z}\{dz - x\,dt - y\,dy - (3x^2 + t)\,dx\}$$

$$= \frac{\partial\phi}{\partial z}\,d\left(z - xt - x^3 - \tfrac{1}{2}y^2\right).$$

Since $d\phi$ and $d(z - xt - x^3 - \frac{1}{2}y^2)$ are perfect differentials, it follows that $\frac{\partial\phi}{\partial z}$ is some function of $z - xt - x^3 - \frac{1}{2}y^2$ alone: and therefore we may effectively take

$$u = z - xt - x^3 - \tfrac{1}{2}y^2 = c$$

as the one integral required.

Ex. 4. Treat similarly the equations

$$\left.\begin{aligned} dx + dy + dz + (x + 1)\,dt &= 0 \\ x\,dx + y\,dy + z\,dz - x\,dt &= 0 \end{aligned}\right\};$$

(Mansion);

* Boole, *Phil. Trans.* 1862, p. 450.

also the equations

$$\left.\begin{aligned}
dx_1 &= \frac{x_2(x_2-x_5)-x_4(x_3+x_4)}{x_2^2-x_4^2}\,dx_2 + \frac{x_2(x_2+x_3)-x_4(x_4-x_5)}{x_2^2-x_4^2}\,dx_4 \\
\tfrac{1}{2}dx_3 &= \frac{x_2x_5+x_3x_4}{x_2^2-x_4^2}\,dx_2 - \frac{x_2x_3+x_4x_5}{x_2^2-x_4^2}\,dx_4 \\
dx_5 &= \frac{x_3-x_5}{x_2+x_4}(dx_2+dx_4)
\end{aligned}\right\}.$$

Ex. 5. The following investigation* is of considerable importance in the theory of partial differential equations of the second order.

In the usual Monge-Boole method of solving the equation

$$R'r+2S's+T't+U'(rt-s^2)=V',$$

where R', S', T', U', V' are functions of x, y, z, p, q, it is assumed that the equation possesses an intermediary integral; and it is known † that, to secure this possession, two conditions must be satisfied by the quantities R', S', T', U', V'. These conditions may be obtained as follows.

In order to form the intermediary integral $u=f(v)$, it is necessary to obtain two integrals $u=a$, $v=b$ (that is, exact integrals in the sense of the preceding paragraphs) of the system of equations ‡

$$\left.\begin{aligned}
U'dy+\lambda_1 T'dx+\lambda_1 U'dp&=0 \\
U'dx+\lambda_2 R'dy+\lambda_2 U'dq&=0 \\
pdx+\quad qdy-\qquad dz&=0
\end{aligned}\right\},$$

where λ_1 and λ_2 are the roots of the equation

$$\lambda^2(R'T'+U'V')+2\lambda U'S'+U'^2=0.$$

For the present purpose we have to assign the conditions that the foregoing system of equations should have two exact integrals.

Writing $\mu=-\dfrac{1}{\lambda}$, $T'=-U'T$, $S'=-U'S$, $R'=-U'R$, $V'=-U'V$ and assuming that U' does not vanish, the equations are

$$\left.\begin{aligned}
dz&=pdx+q\,dy \\
dp&=Tdx+\mu_1 dy \\
dq&=\mu_2 dx+Rdy
\end{aligned}\right\},$$

* It is substantially due to Imschenetsky, "Intégration des équations aux dérivées partielles du second ordre d'une fonction de deux variables," *Grun. Arch.*, t. liv. (1872), especially §§ 13, 14 being pp. 290—314. He has amplified and extended Boole's investigation (l.c. pp. 451, 452): the relation between them will be indicated below. Imschenetsky has however left the conditions (p. 309) in the form of *two equations* involving a dependent variable: without proof, these cannot be taken as merely *two conditions* applying to quantities which do not involve that dependent variable. The explicit conditions are obtained in this investigation.

† See *Treatise*, § 230: the result will evidently apply in the same terms for the case when U does not vanish.

‡ *Treatise*, §§ 232—234.

where μ_1 and μ_2 are the roots of

$$\mu^2 - 2\mu S + RT + V = 0.$$

The partial differential equations associated with the system are

$$\left.\begin{aligned} \Delta\phi &= \left(\frac{\partial}{\partial x} + p\frac{\partial}{\partial z} + T\frac{\partial}{\partial p} + \mu_2\frac{\partial}{\partial q}\right)\phi = 0 \\ \Delta'\phi &= \left(\frac{\partial}{\partial y} + q\frac{\partial}{\partial z} + \mu_1\frac{\partial}{\partial p} + R\frac{\partial}{\partial q}\right)\phi = 0 \end{aligned}\right\},$$

being two equations which involve five variables. Every solution common to the two must also satisfy

$$\begin{aligned} 0 &= (\Delta\Delta' - \Delta'\Delta)\,\phi \\ &= \frac{\partial\phi}{\partial z}(\Delta q - \Delta' p) + \frac{\partial\phi}{\partial p}(\Delta\mu_1 - \Delta' T) + \frac{\partial\phi}{\partial q}(\Delta R - \Delta'\mu_2) \\ &= (\mu_2 - \mu_1)\,\frac{\partial\phi}{\partial z} + \frac{\partial\phi}{\partial p}(\Delta\mu_1 - \Delta' T) + \frac{\partial\phi}{\partial q}(\Delta R - \Delta'\mu_2). \end{aligned}$$

Now Boole (l.c.) investigates the question as to the conditions necessary that the original system may be completely integrable, so as to have three exact integrals. For this purpose, the new equation obtained—being the Jacobian condition to be satisfied by the first two—must vanish identically; hence $\mu_2 = \mu_1 = S$, leading to the condition

$$S^2 = RT + V;$$

and also

$$\Delta S = \Delta' T, \quad \Delta R = \Delta' S,$$

which are the necessary and sufficient conditions. If these be satisfied, then there are three exact integrals, say $u = a$, $v = b$, $w = c$; when p and q are eliminated between them, we have a relation between x, y, z, a, b, c, which is an integral of the original differential equation and can be generalised by Imschenetsky's method*.

Suppose, however, that we assume that the system is incompletely integrable and that it has two exact integrals. Then the new equation must not vanish: and the preceding conditions must therefore not all be satisfied. Assuming that the roots of the quadratic in μ are unequal, and taking

$$\frac{\Delta\mu_1 - \Delta' T}{\mu_2 - \mu_1} = P, \quad \frac{\Delta R - \Delta'\mu_2}{\mu_2 - \mu_1} = Q,$$

the new equation is

$$\Delta''\phi = \left(\frac{\partial}{\partial z} + P\frac{\partial}{\partial p} + Q\frac{\partial}{\partial q}\right)\phi = 0:$$

so that there are now three equations to be satisfied by ϕ. As there are to be two integrals, the system must be complete; and therefore the further equations

$$(\Delta''\Delta - \Delta\Delta'')\,\phi = 0, \quad (\Delta''\Delta' - \Delta'\Delta'')\,\phi = 0$$

* *Treatise*, § 271.

must be satisfied in virtue of the preceding three. When these are formed, they are

$$P\frac{\partial\phi}{\partial z}+\frac{\partial\phi}{\partial p}(\Delta''T-\Delta P)+\frac{\partial\phi}{\partial q}(\Delta''\mu_2-\Delta Q)=0,$$

$$Q\frac{\partial\phi}{\partial z}+\frac{\partial\phi}{\partial p}(\Delta''\mu_1-\Delta'P)+\frac{\partial\phi}{\partial q}(\Delta''R-\Delta'Q)=0;$$

so that the necessary conditions are evidently derivable from a comparison with $\Delta''\phi=0$ alone, and are

$$\Omega_1=-\ P^2+\Delta''T-\Delta P=0,\ \ \Omega_3=-PQ+\Delta''\mu_2-\Delta Q=0,$$

$$\Omega_2=-PQ+\Delta''\mu_1-\Delta'P=0,\ \ \Omega_4=-\ Q^2+\Delta''R-\Delta'Q=0,$$

apparently four in number.

Hence the necessary and sufficient conditions that the differential equation should possess an intermediary integral are that the conditions

$$\Omega_1=0,\ \ \Omega_2=0,\ \ \Omega_3=0,\ \ \Omega_4=0$$

should be satisfied; and these are equivalent to two independent conditions, as they are connected by the relations

$$\left.\begin{array}{l} Q\Omega_1-2P\Omega_2+P\Omega_3=\Delta\,\Omega_2-\Delta'\Omega_1 \\ P\Omega_4-2Q\Omega_3+Q\Omega_2=\Delta'\Omega_3-\Delta\,\Omega_4 \end{array}\right\},$$

which relations are not difficult to obtain.

It has been assumed that the roots of the quadratic in μ are unequal. If they be equal, then it is not difficult to prove that, if the quantities $\Delta S-\Delta'T$ and $\Delta R-\Delta'S$ do not both vanish, the simultaneous system has not more than one exact integral; and some conditions are requisite to secure the possession of one exact integral. For instance, if $\Delta S-\Delta'T$ vanish, then in order that the system may have one exact integral we must have

$$S^2=RT+V$$

as the condition of equality of roots; and S and T must be of the forms

$$S=q\frac{\partial\theta}{\partial z}+\frac{\partial\theta}{\partial y},$$

$$T=p\frac{\partial\theta}{\partial z}+\frac{\partial\theta}{\partial x},$$

where θ is any function of x, y, z.

Similarly it may be shewn that the conditions necessary that the equation

$$r+2Ss+Tt=V$$

should have an intermediary integral (without the subsidiary equations being a completely integrable system) are

$$\Delta''\mu-\Delta\theta=P\theta,\qquad \Delta'\theta=\theta^2,$$

$$\Delta''\nu-\Delta P=P^2,\qquad \Delta''\mu+\Delta'P=P\theta,$$

where

$$\theta=\frac{\Delta'\mu}{\mu-\mu'},\quad P=\frac{\Delta'\nu+\Delta\mu'}{\mu-\mu'},$$

in which μ and μ' denote the roots (supposed to be unequal) of

$$\mu^2-2S\mu+T=0,$$

and Δ, Δ', Δ'' are defined by the equations

$$\Delta\;=\frac{\partial}{\partial x}+\mu\;\frac{\partial}{\partial y}+(p+\mu q)\frac{\partial}{\partial z}+V\frac{\partial}{\partial p}$$

$$\Delta'=\frac{\partial}{\partial q}-\mu'\;\frac{\partial}{\partial p}$$

and

$$\Delta''=(1+q\theta)\frac{\partial}{\partial z}+\theta\frac{\partial}{\partial y}+P\frac{\partial}{\partial p}.$$

The inferences as to the extent of the four conditions and the modifications in case the roots are equal need hardly be stated.

174. The number of exact integrals of the given system of differential equations, supposed to be incompletely integrable, is the number of solutions of the system of associated partial differential equations rendered complete. Methods have already been given for the solution of a complete system of such equations; and they will now therefore be assumed known, say in the forms

$$u_1=c_1,\ u_2=c_2,\ \ldots\ldots,\ u_{n-p}=c_{n-p},$$

the quantities c being constants.

These results being known, they can be used to modify the system of equations. Each of them leads to a differential equation

$$du=0,$$

which is a linear combination of the original equations; and, as the $n-p$ quantities u are functionally independent, the $n-p$ linear combinations are also independent and may therefore be taken as replacing $n-p$ of the original equations appropriately chosen from those which occur in the linear combinations. Now let the variables be transformed, so that $n-p$ of the new $m+n$ variables are $u_1, \ldots, u_{n-p}$; the equations are of the same linear character as before. The first $n-p$ of them are

$$du_1=0,\ \ldots\ldots,\ du_{n-p}=0;$$

the remaining p are linear and involve $u_1, \ldots, u_{n-p}$ with the other $m+p$ variables; and the system so modified is coextensive with the original system.

Now take the $n-p$ exact integrals in the forms

$$u_1 = c_1, \ \ldots\ldots, \ u_{n-p} = c_{n-p},$$

and substitute these in the remaining p equations; we shall then have only a system of p equations in $m+p$ variables alone and constants. Moreover this new system cannot have an exact integral; otherwise, a retransformation to the old variables would lead to another exact integral of the old system.

Hence *an incompletely integrable system of n equations possessing $n-p$ exact integrals can, by means of those integrals, be replaced by a non-integrable system of p equations*; and whatever may be the integral equivalent of the non-integrable system, that integral equivalent combined with the set of exact integrals is the integral equivalent of the incompletely integrable system from which it was derived.

175. By this result and by the remaining question of § 171 we are led to the consideration of the character of the integral equivalent of a non-integrable system of linear equations. Such a system is an obvious generalisation from the case of a single Pfaffian equation: and it is natural to expect that the integral equivalent of such a system will consist of more than the n equations, of which it would be composed were the system completely integrable. As in the case of a single equation, it is desirable to have the integral equivalent of the system as general as possible. This generality will be maintained by two properties: first, the integral equivalent must contain the smallest possible number of integral equations sufficient to lead uniquely to the differential equations, for thus the variations of the variables will be least restricted; second, the equations in such a system must be of as general a form as possible. Three points are thus raised:

(i) the determination of the number of equations in the integral equivalent of a non-integrable system;

(ii) the deduction of some simple integral equivalent of such a system;

(iii) the generalisation of such an integral equivalent when it has been obtained.

Present analysis however seems able to solve only the first of these three problems.

Ex. The system

$$dy_2 = y_3 dy_1, \quad dy_3 = y_4 dy_1,$$

where y_1, y_2, y_3, y_4 are functionally independent, is easily seen to be non-integrable: its integral equivalent must therefore contain more than two equations. One set of integrals is evidently given by

$$y_1 = \text{const.}, \; y_2 = \text{const.}, \; y_3 = \text{const.};$$

another by

$$\begin{aligned} y_4 &= 2a, \text{ a constant,} \\ y_3 &= 2ay_1 + c, \\ y_2 &= \; ay_1^2 + cy_1 + b; \end{aligned}$$

and the most general by

$$\phi(y_1, y_2) = 0,$$

$$\frac{\partial\phi}{\partial y_1} + y_3 \frac{\partial\phi}{\partial y_2} = 0,$$

$$\frac{\partial^2\phi}{\partial y_1^2} + 2y_3 \frac{\partial^2\phi}{\partial y_1 \partial y_2} + y_3^2 \frac{\partial^2\phi}{\partial y_2^2} + y_4 \frac{\partial\phi}{\partial y_2} = 0,$$

where ϕ is any arbitrary function. Each of these sets leads to the two given differential equations, and only to them so far as the given variations are concerned; the geometrical interpretation is obvious.

176. Suppose then we take the system of equations (I), assuming them to be non-integrable: they are

$$\left.\begin{aligned} \Omega_1 &= -dx_{m+1} + A_{11} dx_1 + A_{12} dx_2 + \dots\dots + A_{1m} dx_m = 0 \\ \Omega_2 &= -dx_{m+2} + A_{21} dx_1 + A_{22} dx_2 + \dots\dots + A_{2m} dx_m = 0 \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \Omega_n &= -dx_{m+n} + A_{n1} dx_1 + A_{n2} dx_2 + \dots\dots + A_{nm} dx_m = 0 \end{aligned}\right\} \dots(\text{I}),$$

the coefficients A being functions of the variables.

We first proceed to find the smallest possible number of equations from which there can be constructed an integral equivalent of (I); and for this purpose we use a generalisation of the method adopted by Natani for the similar question in the case of one equation*, viz., the introduction of new variables so chosen as to leave in the equations, when transformed, as few differential elements as possible.

* This question appears to have been solved first by Biermann, "Ueber n simultane Differentialgleichungen der Form $\sum_{\mu=1}^{n+m} X_\mu dx_\mu = 0$," *Schlöm. Zeitschr.*, t. xxx. (1885), pp. 234—244.

Let the new variables be $u_1, u_2, \ldots\ldots, u_p; v_1, v_2, \ldots\ldots, v_r$, where

$$p + r = n + m;$$

it is of course assumed that these variables are functionally independent. Then each of the old variables x is expressible in terms of the new, and so we have

$$\delta x = \Sigma \frac{\partial x}{\partial u_s} \delta u_s + \Sigma \frac{\partial x}{\partial v_t} \delta v_t.$$

Considering these as arbitrary variations, we have

$$\sum_{i=1}^{n} \lambda_i \Omega_i = \sum_{s=1}^{p} \delta u_s \left\{ -\sum_{i=1}^{n} \lambda_i \frac{\partial x_{m+i}}{\partial u_s} + \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_i A_{ij} \frac{\partial x_j}{\partial u_s} \right\}$$
$$+ \sum_{t=1}^{r} \delta v_t \left\{ -\sum_{i=1}^{n} \lambda_i \frac{\partial x_{m+i}}{\partial v_t} + \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_i A_{ij} \frac{\partial x_j}{\partial v_t} \right\},$$

for all quantities λ. When we pass from variations that are arbitrary to such as satisfy the differential equations, the right-hand side of the new equation must vanish. Since the quantities u and v are functionally independent, this can take place owing only to one or other of two causes: either a differential element must vanish or the coefficient of a differential element must vanish. Let then the variables u be those, which have vanishing differential elements, and the variables v be those, the differential elements of which have vanishing coefficients.

The second of these conditions gives, for each of the r values of t, the equation

$$-\sum_{i=1}^{n} \lambda_i \frac{\partial x_{m+i}}{\partial v_t} + \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_i A_{ij} \frac{\partial x_j}{\partial v_t} = 0;$$

and then, for the arbitrary variations, the new form of the above equation is

$$\sum_{i=1}^{n} \lambda_i \Omega_i = \sum_{s=1}^{p} \delta u_s \left\{ -\sum_{i=1}^{n} \lambda_i \frac{\partial x_{m+i}}{\partial u_s} + \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_i A_{ij} \frac{\partial x_j}{\partial u_s} \right\}.$$

Since this equation is valid for all quantities λ, we have

$$\frac{\partial x_{m+i}}{\partial v_t} = \sum_{j=1}^{m} A_{ij} \frac{\partial x_j}{\partial v_t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}$$

for $t = 1, 2, \ldots, r$ and $i = 1, 2, \ldots, n$; and also

$$\Omega_i = \sum_{s=1}^{p} \delta u_s \left\{ -\frac{\partial x_{m+i}}{\partial u_s} + \sum_{j=1}^{m} A_{ij} \frac{\partial x_j}{\partial u_s} \right\}$$

for $i = 1, 2, \ldots, n$. Variations consistent with the differential equations make Ω_i zero: hence we have n equations linear and homogeneous in the p differential elements. Solving these so as to express n of the elements in terms of the remaining q (where $p - n = q$), we have n new equations of the form

$$\left.\begin{aligned} \Upsilon_1 &= -\,du_{q+1} + \sum_{r=1}^{q} U_{1r}\,du_r = 0 \\ \Upsilon_2 &= -\,du_{q+2} + \sum_{r=1}^{q} U_{2r}\,du_r = 0 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \Upsilon_n &= -\,du_{q+n} + \sum_{r=1}^{q} U_{qr}\,du_r = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(\mathrm{I})',$$

which are coextensive with (I), the relations between them being of the form

$$\Omega_i = \sum_{k=1}^{n} \rho_{ik}\Upsilon_k \quad \ldots\ldots\ldots\ldots\ldots(\mathrm{B})$$

for the n values of i, which are $1, 2, \ldots, n$.

Taking any one of the n equations in (B), and comparing the variations of the variable on the two sides of it, we have $m + n$ equations; and these $m + n$ equations involve

(i) the n multipliers $\rho_{i1}, \rho_{i2}, \rho_{i3}, \ldots, \rho_{in}$;
(ii) the p quantities u;
(iii) the nq quantities U.

Eliminating the n multipliers from the $m + n$ equations, we have m equations left involving the p quantities u and the nq quantities U. These m equations survive for each one of the n equations in (B); and therefore we have mn equations left involving the quantities u and U.

Taking the most general case by assuming that these mn equations are independent of one another, we must have

$$mn \leqq p + nq,$$

for otherwise conditions would need to be satisfied; and therefore

$$p \geqq \frac{n}{n+1}(m+n).$$

It thus appears that the least value of p is $\frac{n}{n+1}(m+n)$, which it may be noticed is greater than n in all the cases at present

under consideration: we are taking a system of equations which is not "ordinary", and therefore $m > 1$.

Now, as in the corresponding case of a single equation, the integrals of the system (I) or (I)′ are

$$u_1 = c_1,\ u_2 = c_2,\ \ldots\ldots\ldots,\ u_p = c_p;$$

and we desire to make the number of equations as small as possible. Hence we take the smallest possible value of p.

If then $\dfrac{n}{n+1}(m+n)$ be an integer, we take that integer to be the value of p.

If this quantity be a fraction, we take the next greater integer to be the value of p. Let

$$\frac{n(m+n)}{n+1} = N - \frac{\lambda}{n+1},$$

where λ may be $0, 1, \ldots, n$, according to the value of m; then we take the number of quantities u, that is, the number of equations in the integral equivalent of the differential system, to be N.

177. To render, in the present form, the integral equivalent as general as possible, we shall retain for the quantities u as many arbitrary possibilities as may be, for thereby the variations of the variables will be less limited. Hence all the quantities U will be determined, being

$$n(N-n)$$

in number; and thereafter the mn equations will suffice to determine

$$\begin{aligned} &mn - n(N-n) \\ &= N - \lambda \end{aligned}$$

of the quantities u. Since the total number of the quantities u is N, it follows that λ of them are left undetermined and so may be taken arbitrarily.

Hence we have the theorem:—

The most general integral equivalent of a non-integrable unconditioned system of n Pfaffian equations in m + n variables is composed of N equations of which λ are arbitrary, where N is

equal to $\frac{n(m+n)}{n+1}$, *when this quantity is an integer, and then* λ *is zero; and* N *is the integer next greater than this quantity, when it is not an integer, and then the value of* λ *is given by*

$$\frac{n(m+n)}{n+1} = N - \frac{\lambda}{n+1} \text{ *}.$$

This result agrees with the result obtained by Biermann (l. c.), who uses Natani's method with different analytical details; his result is as follows:—*If*

$$m + n = k(n+1) + \kappa,$$

then there are nk *determinate and* κ *arbitrary integrals,* κ *being less than* $n+1$. In fact

$$\frac{n(m+n)}{n+1} = nk + \frac{n\kappa}{n+1}$$

$$= (nk + \kappa) - \frac{\kappa}{n+1},$$

shewing the identity of the two forms of result.

An important corollary in connection with the value of λ may be inferred. We have

$$m + n = k(n+1) + \lambda$$

and λ is less than $n+1$.

The number λ must be less than m; for if

$$\lambda = m - 1 + \mu,$$

we have

$$n + 1 - \mu = k(n+1),$$

whence k, which is an integer, must be zero unless μ is zero or negative. It therefore follows that *the number of arbitrary integrals must be less than the smaller of the two numbers* m *and* $n+1$.

178. Evidently $m = 1$ gives a system of ordinary simultaneous differential equations; and $n = 1$ gives the former case of a single Pfaffian equation. In the general case, as in the case of a single

* Note that λ must be taken so that the denominator of the fractional part on the right-hand side is $n+1$: the fraction $\frac{n(m+n)}{n+1}$ must not be reduced to lower terms, when such reduction is possible.

equation, it may happen that the system of differential equations may satisfy certain conditions, which will reduce the number of equations in the integral system to be less than N though not in such a way as to leave any such equation exact.

When the system is incompletely integrable, so as to have $n-p$ exact integrals and to be replaced by means of these exact integrals by a non-integrable system of p equations in $m+p$ variables, then the whole number of equations in the system of integrals equivalent to the original differential equations is

$$n-p+\frac{p(m+p)}{p+1},$$

if this quantity be an integer, or, if it be not an integer, the number of integrals is the next greater integer. By taking this quantity in the form

$$n+\frac{p}{p+1}(m-1),$$

we see at once that, *unless a system of n differential equations of the type considered is completely integrable, an integral equivalent contains more than n equations.*

Thus for Ex. 3 of § 173 we have $n=2$, $m=2$. It is there proved that $n-p=1$ for the system of equations; hence it is replaceable by a single equation (non-integrable) in three variables, and the integral of this equation consists of one arbitrary equation and one definite equation, partially dependent on the arbitrary equation. Hence the integral system of the original system contains three equations, viz., one absolutely definite, being exact: one quite arbitrary: one relatively definite, partially dependent on the arbitrary equation.

Ex. We may apply the general result of § 177 to the case of a system of r partial differential equations of the first order involving s independent and r dependent variables, say $x_1, \ldots\ldots, x_s, z^{(1)}, \ldots\ldots, z^{(r)}$. When the r equations are solved for $p_s^{(1)}, \ldots\ldots, p_s^{(r)}$, we have results of the form

$$p_s^{(i)}=p_s^{(i)}(x_1, \ldots\ldots, x_s, z^{(1)}, \ldots\ldots, z^{(r)}, p_1^{(1)}, \ldots\ldots, p_{s-1}^{(r)})=\theta_i;$$

and so we have, as a system of simultaneous Pfaffians, the r equations

$$-dz^{(i)}+p_1^{(i)}dx_1+\ldots\ldots+p_{s-1}^{(i)}dx_{s-1}+\theta_i dx_s=0$$

for $i=1, \ldots\ldots, r$. The number of variables is

$$\begin{array}{ll} r & \text{for the quantities } z, \\ +s & \text{. . . . . } x, \\ +r(s-1) & \text{. . . . . } p, \end{array}$$

and therefore is $rs+s$; hence the number of equations in the equivalent integral system is

$$\frac{r}{r+1}(rs+s),$$

that is, the number is rs. These equations involve the variables z, x and the $r(s-1)$ variables p; when these variables p are eliminated, we have r equations left involving only the variables z and x. These r equations constitute the integral system of the original set of r partial differential equations of the first order.

179. In the case of a single unconditioned Pfaffian in an odd number $2n+1$ of variables, it was proved (§ 69) that the integral system contains a single arbitrary integral and a set of n determinate integrals; and in some of the processes the arbitrary integral is used to remove from the equation one of the variables both in itself and in its differential element. The new equation is an unconditioned Pfaffian in an even number of variables and so its integral system consists solely of determinate integrals.

In the case of a set of unconditioned Pfaffians, a similar use may be made of arbitrary integrals when it is known, as by § 177, that such integrals occur in the integral system. If there be λ arbitrary integrals, they can be used to remove from the equations λ of the variables occurring in themselves and in their differential elements. And this is the only way in which such integrals can be used in modification of the system of equations: they cannot be used to diminish the number of equations, for such a result would imply either that an exact integral could be framed, a conclusion excluded by our initial hypothesis, or that an equation became evanescent owing to relations derived from those integral equations when they were differentiated.

That the latter is impossible is an immediate inference from the result of § 177 (*fin.*) that the number of arbitrary integrals is less than the smaller of the two integers m and $n+1$. Since each equation in the original system (I) contains $m+1$ differential elements, it cannot be made evanescent by means of a number of equations of the form

$$d\phi = 0,$$

which are in number less than either m or $n+1$.

Hence the λ arbitrary equations in the integral system may be used to eliminate λ of the variables from the set of differential

equations; and the transformed equivalent system consists still of n members and involves $m+n-\lambda$ variables. Since $m+n-\lambda=k(n+1)$, the integral, equivalent to the new system, contains nk determinate equations and no arbitrary equations; that is, *the arbitrary integrals can be used to transform the system of differential equations to a new system, the integral equivalent of which is composed entirely of determinate equations.*

If then any given system possess arbitrary integrals, we shall suppose that this transformation is effected.

180. Care must be exercised in particular cases. Thus in Ex. 1, § 173, the value of m is 4 and is not 2, although only two differential elements occur on the right-hand side; the fact is that dx_6 and dx_7 must be considered as occurring on each right-hand side with a zero coefficient.

Also, in connection with such an equation, it is desirable to assign arbitrary integrals in such a way that they may lead to determinate equations of a general type and not of a type corresponding to singular solutions in ordinary equations. Thus if, as the three arbitrary equations for the quoted example, we take

$$\phi(x_1, x_2, x_3)=a, \quad \psi(x_1, x_2, x_3)=\beta$$

with some other, which for the present remarks need not be specified, then the first equation

$$dx_3=x_4dx_1+x_5dx_2$$

will be satisfied either by a relation

$$\begin{vmatrix} \frac{\partial\phi}{\partial x_1}, & \frac{\partial\phi}{\partial x_2}, & \frac{\partial\phi}{\partial x_3} \\ \frac{\partial\psi}{\partial x_1}, & \frac{\partial\psi}{\partial x_2}, & \frac{\partial\psi}{\partial x_3} \\ x_4, & x_5, & -1 \end{vmatrix}=0,$$

which is of the type of a singular solution for it contains no new arbitrary element; or by

$$x_3=\text{constant},$$

which, with the other two, gives also

$$x_1=\text{constant}, \quad x_2=\text{constant},$$

as replacing the two former equations. And then, in connection with the latter, we have from the other two

$$x_4=\text{constant}, \quad x_5=\text{constant}.$$

The total number of these integrals is thus five, being one less than the number to be expected according to § 177; but they constitute a set of very limited variations, and they are independent of x_6 and x_7, so that they can hardly be entitled to rank as more than a special set of integrals.

The set of equations just considered are, when expressed in ordinary notation,

$$\left.\begin{aligned} dz &= pdx + qdy \\ dp &= r\,dx + \theta dy \\ dq &= \theta\,dx + t\,dy \end{aligned}\right\},$$

the set subsidiary to

$$s = \theta = \text{function of } x,\ y,\ z,\ p,\ q,\ r,\ t,$$

a partial equation of the second order with two independent variables. The theory indicates that the integral equivalent consists of a set of six equations, three of which are arbitrary; and these equations involve the seven variables x, y, z, p, q, r, t. When from the six equations four quantities (say p, q, r, t) are eliminated, there remain two equations in z, x, y alone, which two equations are the integral equivalent of the equation

$$s = \theta\,(x,\ y,\ z,\ p,\ q,\ r,\ t);$$

that is, the most general partial differential equation of the second order in two independent variables has two equations for its integral equivalent.

Geometrically interpreted, this result is that the original differential equation $s = \theta$ represents some property of a surface at points which lie on its line of contact with another surface having the same curvature at such points; and the character of each of these surfaces affects that of the other.

There are generally six equations in the integral equivalent, for the system possesses no exact integral (Ex. 1, § 173). But it may happen that, for particular forms of θ, one of the arbitrary equations is isolated, so that it does not affect the forms of the determinate equations. Then the result of elimination is to leave one equation, after p, q, r, t are eliminated from the other five, and this equation involves z, x, y; in its form it is affected by the arbitrary characters of two of the assumed integrals and so there will be two arbitrary elements in it, but it is entirely independent of the remaining arbitrary equation which, when substitution is made for p, q, r, t, comes to be an equation in z, x, y and so is the equation of a new surface with an entirely arbitrary formal element. The integral system really consists of the two equations; but the second is an isolated equation and the differential equation is satisfied by the first alone.

Geometrically interpreted, this result is that the original differential equation $s = \theta$ represents some property of the surface, determined by the one integral, along all curves, which are its lines of contact with the arbitrary surface determined by the other integral. As the latter is by its arbitrary character independent of the former, the curves of contact are all that can be drawn on the former surface; and thus the indicated property may be regarded as a property of the whole of that surface.

Hence, according to the foregoing theory, the following would be the

solution of the partial differential equation of the second order. The subsidiary equations are

$$\left.\begin{aligned} dz &= pdx + qdy + 0dr + 0dt \\ dp &= rdx + \theta dy + 0dr + 0dt \\ dq &= \theta dx + tdy + 0dr + 0dt \end{aligned}\right\},$$

where θ is a function of the quantities z, p, q, x, y, r, t determined by the given differential equation. There are three arbitrary integrals of the system: let them be

$$\left.\begin{aligned} r &= f_1(x, z, p, q) \\ t &= f_2(x, z, p, q) \\ y &= f_3(x, z, p, q) \end{aligned}\right\}.$$

When these are substituted the equations become a system, the integrals of which are determinate; when they are solved, they take the forms

$$dx = \frac{dz}{Z} = \frac{dp}{P} = \frac{dq}{Q},$$

where P, Q Z are functions of x, z, p, q and involve the functional forms of f_1, f_2, f_3. When these are integrated, their integrals are of the form

$$\left.\begin{aligned} g_1(x, p, q, z) &= a \\ g_2(x, p, q, z) &= b \\ g_3(x, p, q, z) &= c \end{aligned}\right\},$$

the elimination of p and q from which and from

$$f_3(x, p, q, z) = y$$

will leave two integrals, being the necessary number.

A similar theory will apply to partial differential equations of other orders, and also to a system of two (or more) partial differential equations of the second order*. We then have as subsidiary equations

$$\left.\begin{aligned} dz &= pdx + q\,dy + 0ds \\ dp &= \theta\,dx + s\,dy + 0ds \\ dq &= s\,dx + \phi dy + 0ds \end{aligned}\right\},$$

a system with three determinate and two arbitrary equations in its integral system in general.

181. Biermann (l. c. § 176) declares that Pfaff's method of integration—a process of successive reduction—cannot be used for an unconditioned system; and he shews that Clebsch's second method will not apply, by taking (I)′ as the equivalent of (I) and proving that a process similar to Clebsch's does not lead to

* The cases in which the equations are of the complete type have been considered by Vályi, *Crelle*, t. xcv., pp. 99—101.

equations for u that are the same as the equations for the ratios of the quantities U. This is naturally to be expected, for there are $n^2(k-1)$ quantities U and only $m+n-nk$ independent variables other than u in the entirely transformed system, and in the cases at present under consideration ($n>1$ and $m>1$) it is easy to see that $n^2(k-1)-1$ is greater than $m+n-nk$.

The following process, as an attempt to effect a reduction, is a generalisation of Natani's process for the case of a single Pfaffian; it suffices to shew, as is remarked at the end of the investigation, that the generalisation is not effective for the desired purpose.

Consider the series of equations (I) and, after the explanations of § 179, assume that all the integrals are determinate, so that $m+n$ is divisible by $n+1$; let $m+n=p+1$.

Then in any transformation of the equations it will be necessary to take $p+1$ new variables; let these be $u_1, \ldots, u_p, v$. As desiderata of the new transformation, suppose first that each of the expressions $\Omega_1, \Omega_2, \ldots, \Omega_n$ is independent of dv; the subsidiary equations are

$$-\frac{\partial x_{m+i}}{\partial v}+A_{i1}\frac{\partial x_1}{\partial v}+A_{i2}\frac{\partial x_2}{\partial v}+\ldots\ldots+A_{im}\frac{\partial x_m}{\partial v}=0\ldots(1),$$

for $i=1, \ldots, n$. Then it will follow that any linear combination of the transformed expressions Ω is independent of the differential element dv.

Now take such a linear combination

$$\lambda_1\Omega_1+\lambda_2\Omega_2+\ldots+\lambda_n\Omega_n;$$

it does not involve dv; suppose that, if possible, the coefficients λ (which are variable quantities) are so determined that the new expression is independent of v also. This requires that, for arbitrary variations of the variables, the equation

$$\frac{\partial}{\partial v}\left(\sum_{i=1}^{n}\lambda_i\Omega_i\right)=0$$

should be satisfied; and therefore

$$-\sum_{i=1}^{n}\delta x_{m+i}\frac{\partial\lambda_i}{\partial v}-\sum_{i=1}^{n}\lambda_i\frac{\partial(\delta x_{m+i})}{\partial v}$$
$$+\sum_{i=1}^{n}\frac{\partial\lambda_i}{\partial v}\sum_{s=1}^{m}A_{is}\delta x_s+\sum_{i=1}^{n}\lambda_i\sum_{s=1}^{n}A_{is}\frac{\partial(\delta x_s)}{\partial v}+\sum_{i=1}^{n}\lambda_i\sum_{s=1}^{n}\frac{\partial A_{is}}{\partial v}\delta x_s=0.$$

Now, when in (1) substitution for the quantities x in terms of $u_1, \ldots, u_p, v$ is effected, the result is an identity; and therefore any arbitrary variation of the left-hand side is zero. Hence for each value of i

$$-\delta\frac{\partial x_{m+i}}{\partial v}+\sum_{s=1}^{m} A_{is}\delta\frac{\partial x_s}{\partial v}+\sum_{s=1}^{m}\frac{\partial x_s}{\partial v}\delta A_{is}=0.$$

Multiplying this by λ_i, summing for all the values of i, and remembering that, as δ implies an arbitrary variation,

$$\frac{\partial}{\partial v}(\delta x)=\delta\frac{\partial x}{\partial v},$$

we have, from a comparison of the parts of the two equations which involve the quantities $\frac{\partial}{\partial v}(\delta x)$, the relation

$$-\sum_{i=1}^{n}\delta x_{m+i}\frac{\partial \lambda_i}{\partial v}+\sum_{i=1}^{n}\frac{\partial \lambda_i}{\partial v}\sum_{s=1}^{m}A_{is}\delta x_s+\sum_{i=1}^{n}\lambda_i\sum_{s=1}^{m}\left\{\frac{\partial A_{is}}{\partial v}\delta x_s-\frac{\partial x_s}{\partial v}\delta A_{is}\right\}=0$$

satisfied for arbitrary variations δ. Now

$$\frac{\partial A_{is}}{\partial v}=\sum_{\mu=1}^{m+n}\frac{\partial A_{is}}{\partial x_\mu}\frac{\partial x_\mu}{\partial v},$$

$$\delta A_{is}=\sum_{\mu=1}^{m+n}\frac{\partial A_{is}}{\partial x_\mu}\delta x_\mu;$$

and therefore

$$-\sum_{i=1}^{n}\delta x_{m+i}\left\{\frac{\partial \lambda_i}{\partial v}+\sum_{r=1}^{n}\lambda_r\sum_{s=1}^{m}\frac{\partial x_s}{\partial v}\frac{\partial A_{rs}}{\partial x_{m+i}}\right\}$$
$$+\sum_{s=1}^{m}\delta x_s\left\{\sum_{r=1}^{n}\left(\frac{\partial \lambda_r}{\partial v}A_{rs}+\lambda_r\sum_{i=1}^{n}\frac{\partial A_{rs}}{\partial x_{m+i}}\frac{\partial x_{m+i}}{\partial v}+\lambda_r\sum_{t=1}^{m}c_{rst}\frac{\partial x_t}{\partial v}\right)\right\}=0,$$

where

$$c_{rst}=\frac{\partial A_{rs}}{\partial x_t}-\frac{\partial A_{rt}}{\partial x_s}.$$

Since this equation is to hold for arbitrary variations of the variables, the coefficients of the separate variations must vanish; and therefore

$$0=\frac{\partial \lambda_i}{\partial v}+\sum_{r=1}^{n}\lambda_r\left(\sum_{s=1}^{m}\frac{\partial A_{rs}}{\partial x_{m+i}}\frac{\partial x_s}{\partial v}\right)\quad\ldots\ldots\ldots\ldots\ldots(2)$$

for $i=1, \ldots, n$ and

$$0=\sum_{r=1}^{n}\left[A_{rs}\frac{\partial \lambda_r}{\partial v}+\lambda_r\left\{\sum_{i=1}^{n}\frac{\partial A_{rs}}{\partial x_{m+i}}\frac{\partial x_{m+i}}{\partial v}+\sum_{t=1}^{m}c_{rst}\frac{\partial x_t}{\partial v}\right\}\right.\quad\ldots(3)$$

for $s=1, \ldots, m$. There are therefore $m+n$ equations.

We now have in (1), (2), (3) a total of $m+2n$ equations to determine the $m+n$ quantities x as functions of v and the n coefficients λ. Each of the equations is linear in the derivatives of $m+2n$ quantities with regard to v, and so it would at first appear as if elimination would at once lead to one relation between the quantities λ and the variables x which would be obtainable without any integration whatever.

The explanation of this point is that the set of $m+2n$ equations is subject to one linear relation, which is easily seen to be

$$\sum_{s=1}^{m} \Phi_s \frac{\partial x_s}{\partial v} = \sum_{r=1}^{n} \left(\Theta_r \frac{\partial x_{m+r}}{\partial v} + \Omega'_r \frac{\partial \lambda_r}{\partial v} \right),$$

where $\Omega'_r = 0$, $\Theta_r = 0$, $\Phi_s = 0$ are respectively the equations (1), the equations (2) and the equations (3). Hence the whole system of equations is equivalent to $m+2n-1$ independent equations, linearly homogeneous in $m+2n$ quantities to be determined.

Taking new quantities $\theta_2, \ldots, \theta_n$ defined by the equations

$$\lambda_r = \theta_r \lambda_1, \quad (r = 2, \ldots, n),$$

and assuming the equations to be soluble, their solutions can be expressed in the forms

$$\frac{\frac{\partial x_1}{\partial v}}{X_1} = \frac{\frac{\partial x_2}{\partial v}}{X_2} = \ldots\ldots = \frac{\frac{\partial x_m}{\partial v}}{X_m} = \frac{\frac{\partial x_{m+1}}{\partial v}}{Y_1} = \ldots\ldots = \frac{\frac{\partial x_{m+n}}{\partial v}}{Y_n}$$

$$= \frac{\frac{1}{\lambda_1}\frac{\partial \lambda_1}{\partial v}}{P_1} = \frac{\frac{1}{\lambda_1}\frac{\partial \lambda_2}{\partial v}}{P_2} = \ldots\ldots = \frac{\frac{1}{\lambda_1}\frac{\partial \lambda_n}{\partial v}}{P_n},$$

where $X_1, \ldots, X_m, Y_1, \ldots, Y_n, P_1, \ldots, P_n$ are functions of the variables x and of the $n-1$ variables θ alone.

The system must first be changed so as to give equations involving the variables to be determined. We have

$$\frac{1}{\lambda_1}\frac{\partial \lambda_2}{\partial v} = \frac{\partial \theta_r}{\partial v} + \theta_r \frac{1}{\lambda_1}\frac{\partial \lambda_1}{\partial v},$$

and so we obtain the transformed system

$$\frac{\frac{\partial x_1}{\partial v}}{X_1} = \ldots\ldots = \frac{\frac{\partial x_m}{\partial v}}{X_m} = \frac{\frac{\partial x_{m+1}}{\partial v}}{Y_1} = \ldots\ldots = \frac{\frac{\partial x_{m+n}}{\partial v}}{Y_n}$$

$$= \frac{\frac{\partial \theta_2}{\partial v}}{P_2 - \theta_2 P_1} = \frac{\frac{\partial \theta_3}{\partial v}}{P_3 - \theta_3 P_1} = \ldots\ldots = \frac{\frac{\partial \theta_n}{\partial v}}{P_n - \theta_n P_1},$$

a system of $(m+n)+(n-1)-1$ equations.

First, we find the $n-1$ quantities θ as functions of the variables x; their expressions will involve an aggregate of $n-1$ arbitrary independent constants. When values thence derived are substituted in the first $m+n-1$ equations, the latter come to be equations in the $m+n$ variables x alone; and as in the corresponding case of a single Pfaffian equation treated by Natani's process, the $m+n-1$ integrals of these equations are the new variables $u_1, \ldots, u_p$, which are $p(=m+n-1)$ independent functions of $x_1, \ldots, x_{m+n}$.

So far as concerns the remaining variable v, the sole condition which applies to it is that it must be a function of the variables x independent of $u_1, \ldots, u_p$. Subject to this condition we may choose it at will and so we may take $v = x_1$, an assumption similar to Natani's for the corresponding case; the value of λ_1 is then determined by a quadrature and the values of $\lambda_2, \ldots, \lambda_n$ are thence inferred through the values of $\theta_2, \ldots, \theta_n$.

Ex. 1. The equations subsidiary to the system

$$\left.\begin{aligned} \Omega &= dy + \sum_{s=1}^{n} Y_s dx_s = 0 \\ \Upsilon &= dz + \sum_{s=1}^{n} Z_s dx_s = 0 \end{aligned}\right\}$$

(being the conditions for transformation such that Ω and Υ are both free from dv and that $\lambda\Omega + \mu\Upsilon$ is free also from v) are

$$\frac{\partial \lambda}{\partial v} = \lambda \sum_{i=1}^{n} \frac{\partial Y_i}{\partial y}\frac{\partial x_i}{\partial v} + \mu \sum_{i=1}^{n} \frac{\partial Z_i}{\partial y}\frac{\partial x_i}{\partial v}$$

$$\frac{\partial \mu}{\partial v} = \lambda \sum_{i=1}^{n} \frac{\partial Y_i}{\partial z}\frac{\partial x_i}{\partial v} + \mu \sum_{i=1}^{n} \frac{\partial Z_i}{\partial z}\frac{\partial x_i}{\partial v}$$

$$0 = \frac{\partial y}{\partial v} + \sum_{s=1}^{n} Y_s \frac{\partial x_s}{\partial v}$$

$$0 = \frac{\partial z}{\partial v} + \sum_{s=1}^{n} Z_s \frac{\partial x_s}{\partial v}$$

$$0 = Y_s \frac{\partial \lambda}{\partial v} + Z_s \frac{\partial \mu}{\partial v} + \frac{\partial y}{\partial v}\left(\lambda \frac{\partial Y_s}{\partial y} + \mu \frac{\partial Z_s}{\partial y}\right) + \frac{\partial z}{\partial v}\left(\lambda \frac{\partial Y_s}{\partial z} + \mu \frac{\partial Z_s}{\partial z}\right)$$
$$+ \sum_{i=1}^{n} (\lambda y_{si} + \mu z_{si}) \frac{\partial x_i}{\partial v},$$

where the last equation holds for $s=1, \ldots, n$ and

$$y_{si}=\frac{\partial Y_s}{\partial x_i}-\frac{\partial Y_i}{\partial x_s}, \quad z_{si}=\frac{\partial Z_s}{\partial x_i}-\frac{\partial Z_i}{\partial x_s}.$$

The system of $n+4$ equations consists of a set of only $n+3$ independent equations.

Ex. 2. Let the two equations

$$\begin{aligned} \Omega &= -dx_1+a_3dx_3+a_4dx_4=0 \\ \Upsilon &= -dx_2+\beta_3dx_3+\beta_4dx_4=0 \end{aligned}$$

be unconditioned.

Then, taking $\lambda(\Omega+\theta\Upsilon)$ as in the text and transforming to new variables u_1, u_2, u_3, v, so that neither v nor dv occurs in the transformed expression for $\lambda(\Omega+\theta\Upsilon)$, we have a result of the form

$$\lambda(\Omega+\theta\Upsilon)=U_1du_1+U_2du_2+U_3du_3,$$

in which U_1, U_2, U_3 are functions of u_1, u_2, u_3 alone. Now the reduced form of the right-hand side is (§§ 126, 144)

$$dy_2+y_3dy_1,$$

where y_1, y_2, y_3 are functions of u_1, u_2, u_3 and therefore are integrals of the subsidiary system, so that y_1, y_2, y_3 and v may be taken as new variables. Moreover the integrals of the subsidiary system are such that, when they are substituted into Υ, the new expression is of the form

$$V_1dy_1+V_2dy_2+V_3dy_3,$$

where the coefficients V are functions of y_1, y_2, y_3 and v. Hence the original equations are coextensive with

$$\begin{aligned} dy_2+y_3dy_1&=0, \\ V_1dy_1+V_2dy_2+V_3dy_3&=0. \end{aligned}$$

The second of these can, by means of the first, be changed to

$$dy_3+y_4dy_1=0,$$

where

$$y_4=\frac{V_1-V_2y_3}{V_3}.$$

Now as v is at our choice, subject to the sole condition that it is functionally independent of y_1, y_2, y_3, we may take, as the fourth variable for the transformation, any function of v, y_1, y_2, y_3, which is not independent of v; and so we may take y_4 as the fourth variable.

Hence the original equations can be replaced by the two equations

$$\left.\begin{aligned} dy_2+y_3dy_1&=0 \\ dy_3+y_4dy_1&=0 \end{aligned}\right\}.$$

This result, differently obtained and by different considerations, was first given by Engel *. He also enunciated the following result, which is practically a résumé of results which are alternative:

* "Zur Invariantentheorie der Systeme von Pfaff'schen Gleichungen", *Leips. Sitzungsb.*, (1889), pp. 157—176.

The system of equations

$$\left.\begin{aligned} -dx_1+a_3dx_3+a_4dx_4=0 \\ -dx_2+\beta_3dx_3+\beta_4dx_4=0 \end{aligned}\right\},$$

where the quantities a and β are functions of x_1, x_2, x_3, x_4, can be transformed to one or other of the pairs of equations

$$\left.\begin{aligned} dy_1=0 \\ dy_2=0 \end{aligned}\right\}; \quad \left.\begin{aligned} dy_2-y_3dy_1=0 \\ dy_4=0 \end{aligned}\right\}; \quad \left.\begin{aligned} dy_2-y_3dy_1=0 \\ dy_3-y_4dy_1=0 \end{aligned}\right\}.$$

The integration of the transformed equations in the third of these cases, the one treated above, has already been discussed in the Example in § 175.

182. Suppose now that the subsidiary equations are integrated to determine in the first place the quantities θ, which will be effected by equations of the form

$$\left.\begin{aligned} a_1 &= h_1 \;(x_1, \ldots, x_{m+n}, \theta_2, \ldots, \theta_n) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n-1} &= h_{n-1}(x_1, \ldots, x_{m+n}, \theta_2, \ldots, \theta_n) \end{aligned}\right\} \ldots\ldots\ldots\ldots(4).$$

When the values of θ hence derived are substituted in the first $m+n-1$ equations, so that they come to be equations in the variables x only, their integrals take the form

$$\left.\begin{aligned} u_1 &= g_1\,(x_1, \ldots, x_{m+n}, a_1, \ldots, a_{n-1}) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ u_p &= g_p\,(x_1, \ldots, x_{m+n}, a_1, \ldots, a_{n-1}) \end{aligned}\right\} \ldots\ldots\ldots\ldots(5);$$

and then, when λ has been determined, we have the equation

$$\lambda_1\,(\Omega_1+\theta_2\,\Omega_2+\ \ldots\ldots\ +\theta_n\,\Omega_n)=\lambda_1\,\Omega$$

expressible in the form

$$\lambda_1\,\Omega = U_1\,du_1 + U_2\,du_2 + \ \ldots\ldots\ + U_p\,du_p,$$

where the coefficients U are functions of the variables u alone.

The right-hand side implicitly contains the constants $a_1, \ldots, a_{n-1}$, which are left undetermined by the subsidiary equations. Hence when another set of constants is taken, a different set of quantities θ will occur and therefore a new combination of the equations (I) will arise; and it is evident that n different sets of constants will lead to n independent combinations of the equations (I) and so will lead to a system of n equations equivalent to (I).

In general, however, a new set of constants in (4) will lead to different expressions for the quantities θ; and so when these are

substituted in the subsidiary equations they will vary in form, and therefore the quantities u will change. Hence in general the new system of equations will not contain throughout the same variables.

183. But now suppose that $a_1, \ldots, a_{n-1}$ are a given set of constants, and let them be subjected to infinitesimal variational increments $\alpha_1, \ldots, \alpha_{n-1}$. Then there will be new multipliers $\theta_2+I_2, \ldots, \theta_n+I_n$, where the infinitesimal quantities I are determined by the $n-1$ equations

$$\alpha_i = \sum_{r=2}^{n} I_r \frac{\partial h_i}{\partial \theta_r}$$

for $i=1, \ldots, n-1$; and there will be new variables $u_1', \ldots, u_p'$ given by

$$u_r' - u_r = \sum_{i=1}^{n-1} \alpha_i \frac{\partial g_r}{\partial a_i}.$$

If then all the quantities $\frac{\partial g_r}{\partial a_i}$, *for the* p *values of* r *and the* $n-1$ *values of* i, *can be expressed as functions of* $u_1, \ldots, u_p$ *and the constants* a (*now regarded as a determinate set*), *then* u_r' *is a function of the* p *quantities* $u_1, \ldots, u_p$; and the conditions that must be satisfied to ensure the justification of this hypothesis are the conditions that the $m+n$ variables x can be eliminated from the $m+n$ equations involving $u_1, \ldots, u_p, \frac{\partial g}{\partial a}$.

It is easy to prove that, if $A_1, \ldots, A_{m+n}$ be the expressions for $X_1, \ldots, X_m, Y_1, \ldots, Y_n$ when in the latter the values of the quantities θ in terms of the variables x and the constants a are substituted, then the conditions necessary and sufficient to justify the foregoing hypothesis are that the equations

$$\frac{1}{A_1}\frac{\partial A_1}{\partial a} = \frac{1}{A_2}\frac{\partial A_2}{\partial a} = \ldots\ldots = \frac{1}{A_{m+n}}\frac{\partial A_{m+n}}{\partial a}$$

shall be satisfied for all the values a. These require that the constants a shall enter into $A_1, A_2, \ldots, A_{m+n}$ only by occurrence in a factor common to all. In that case the differential equation satisfied by $u_1, \ldots, u_p$, viz. that which is associated with the subsidiary equations of § 181 in the form

$$A_1\frac{\partial u}{\partial x_1} + A_2\frac{\partial u}{\partial x_2} + \ldots\ldots + A_{m+n}\frac{\partial u}{\partial x_{m+n}} = 0,$$

is an equation which, on removal of the factor common to the A's, is independent of the constants a. Hence all the functions g, which are its solutions, are independent of the a's; and so $u'=u$ and the transformed system will contain throughout the same variables.

The system will therefore take the form

$$\left.\begin{array}{l} U_1^{(1)}du_1 + \ldots\ldots + U_p^{(1)}du_p = 0 \\ U_1^{(2)}du_1 + \ldots\ldots + U_p^{(2)}du_p = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ U_1^{(n)}du_1 + \ldots\ldots + U_p^{(n)}du_p = 0 \end{array}\right\},$$

where all the coefficients U are functions of the variables u. Now, since

$$m+n=k(n+1),$$

we have

$$p=m+n-1=(k-1)(n+1)+n,$$

that is, the new system has n arbitrary integrals and $(k-1)n$ determinate integrals. The former may be taken to be $u_1=\text{constant}, \ldots\ldots, u_n=\text{constant}$; and the new system is changed to one in which, with $m-1$ variables only, there are n equations—that is, substantially the same case as that already treated.

The cases in which the conditions just specified are actually satisfied are those which arise only for very special forms of the coefficients in the original system and therefore only for a very limited number of cases; and hence it is to be inferred that a system of simultaneous unconditioned Pfaffians cannot be integrated by what is the natural generalisation of Natani's method.

The partial differential equations, next in order of simplicity and of interest after partial differential equations of the first order in a single dependent variable, are simultaneous partial differential equations of the first order involving two dependent variables. The subsidiary systems, associated with these, do not satisfy the conditions indicated; and therefore the equations cannot be integrated by the foregoing method.

184. Grassmann* has shewn that the integration of a partial differential equation of any order can be effected after the integration of the equation $Xdx=0$, where Xdx is now extensive and not merely numerical as in Chapter V: and this will apply to the integration of a system of unconditioned Pfaffians which include, as a special case, the equations subsidiary to a partial differential equation of order higher than the first. But beyond proving the relation

$$Xdx = U_1du_1 + U_2du_2 + \ldots,$$

where $u_1=\text{constant}$, $u_2=\text{constant}$, ... are the integral system of the set of equations and the coefficients U are no longer numerical, he has made no actual contribution to the solution, probably because the earlier methods used for Pfaff's problem are no longer applicable.

185. And so the solution of the problem of obtaining the integral equivalent of a simultaneous system of unconditioned

* *Ausdehnungslehre* (edition of 1862), § 501.

Pfaffians does not appear possible by any methods at present known which are effective for the case of a single Pfaffian. It is, in fact, one of the most general problems of the integral calculus; the discovery of its solution lies in the future.

186. Such then is the present position of the second of the three problems stated in § 175. The third of those problems, viz., the generalisation of a given solution, is (except for a few particular cases such as the example of § 175) still unsolved; and it appears from the following simple illustration that the generalisation must remain unsolved until the second of the problems there indicated is solved.

The number of equations in the integral system equivalent to a pair of unconditioned Pfaffians in six variables is, by § 176, four; let them be

$$a=\text{constant},\quad b=\text{constant},\quad e=\text{constant},\quad f=\text{constant},$$

so that, as there, the reduced equivalent set of differential equations is

$$\left.\begin{aligned} de&=Ada+Bdb\\ df&=Cda+Ddb\end{aligned}\right\},$$

where A, B, C, D are functions of a, b, e, f and of the two other new independent variables, say x and y, necessary for the expression of the original six. Then in order to generalise the particular set of solutions it is necessary to obtain the equations which make it possible to pass from the above pair to

$$\left.\begin{aligned} dr&=Pdp+Qdq\\ ds&=Rdp+Sdq\end{aligned}\right\},$$

where p, q, r, s are functions of a, b, e, f, x, y and P, Q, R, S are derivable by mere substitution when these functions are known.

That the equations may be coextensive, we must have for some values of ρ, σ, ρ', σ' the relations

$$-dr+Pdp+Qdq=\rho\,(-de+Ada+Bdb)+\sigma\,(-df+Cda+Dde)$$

$$-ds+Rdp+Sdq=\rho'\,(-de+Ada+Bdb)+\sigma'\,(-df+Cda+Dde)$$

identically satisfied for proper values of p, q, r, s, P, Q, R, S. Each of these relations gives six equations. Eliminating ρ, σ, P, Q from the first set of six, we have

$$\left.\begin{aligned} \frac{\partial(p,q,r)}{\partial(x,y,a)}+A\frac{\partial(p,q,r)}{\partial(x,y,e)}+C\frac{\partial(p,q,r)}{\partial(x,y,f)}&=0\\ \frac{\partial(p,q,r)}{\partial(x,y,b)}+B\frac{\partial(p,q,r)}{\partial(x,y,e)}+D\frac{\partial(p,q,r)}{\partial(x,y,f)}&=0\end{aligned}\right\},$$

and similarly from the second set of six we have

$$\left.\begin{aligned}\frac{\partial(p, q, s)}{\partial(x, y, a)} + A\frac{\partial(p, q, s)}{\partial(x, y, e)} + C\frac{\partial(p, q, s)}{\partial(x, y, f)} = 0 \\ \frac{\partial(p, q, s)}{\partial(x, y, b)} + B\frac{\partial(p, q, s)}{\partial(x, y, e)} + D\frac{\partial(p, q, s)}{\partial(x, y, f)} = 0\end{aligned}\right\};$$

altogether a set of four simultaneous partial differential equations of the first order determining four dependent variables *.

Unless the dependent variables in these equations can be partially separated, the system remains merely in its most general form; and the solution of the system then depends on that of a system of simultaneous partial differential equations in several dependent variables and therefore on that of a system of simultaneous Pfaffians, a problem which hitherto has defied solution. The natural method of attempting the partial separation of the dependent variables is the generalisation of Mayer's method (§ 134) in the theory of tangential transformations; we take

$$\frac{d}{da} = \frac{\partial}{\partial a} + A\frac{\partial}{\partial e} + C\frac{\partial}{\partial f}, \quad \frac{d}{db} = \frac{\partial}{\partial b} + B\frac{\partial}{\partial e} + D\frac{\partial}{\partial f},$$

so that the equations become

$$\left.\begin{aligned}\frac{dr}{da} = P\frac{dp}{da} + Q\frac{dq}{da} \\ \frac{dr}{db} = P\frac{dp}{db} + Q\frac{dq}{db}\end{aligned}\right\}$$

together with

$$\left.\begin{aligned}\frac{\partial r}{\partial x} = P\frac{\partial p}{\partial x} + Q\frac{\partial q}{\partial x} \\ \frac{\partial r}{\partial y} = P\frac{\partial p}{\partial y} + Q\frac{\partial q}{\partial y}\end{aligned}\right\},$$

* The solution of n simultaneous partial differential equations of the first order in n dependent and two independent variables has been effected by Hamburger in his memoir "Zur Theorie der Integration eines Systems von n *nicht linearen* partiellen Differentialgleichungen erster Ordnung mit zwei unabhängigen und n abhängigen Veränderlichen", *Crelle*, t. xciii. (1882), pp. 188—214. But his method applies only to unconditioned equations when the number of independent variables is two. When the number m of independent variables is greater than two and the n simultaneous equations involve n dependent variables, then certain conditions must be satisfied in order that his method may apply; and it may be proved that the number of these conditions, independent of one another, is $(n-1)(m-2)$.

Other investigations (e.g. Kowalevski, *Crelle*, t. lxxx. (1875), pp. 1—32) relate chiefly to the proof of the existence of solutions of such systems by obtaining them in the form of converging series; but no process of integration is given except in the form of series containing arbitrary initial values of the variables.

which are all the equations obtainable, free from λ and μ. On carrying out his method it appears (I do not reproduce the work) that the resulting equations for P, Q, p, q are not of the form required; and so the corresponding partial separation is not possible.

Hence we make the inference in the text—that the generalisation of a given particular set of solutions is not at present possible, as it depends upon the possibility of solution of the second problem, for which there is at present no effective method.

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